Fundamentals of Model Predictive Control

Preliminaries

Antonio Ferramosca

Università degli Studi di Bergamo

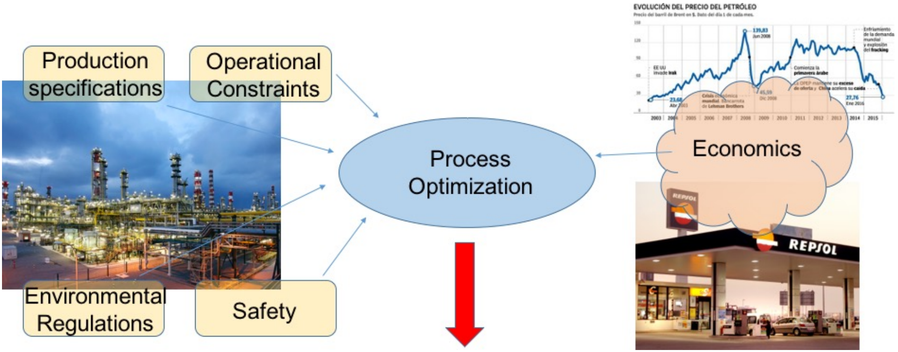
June 2022

Outline

A motivational example

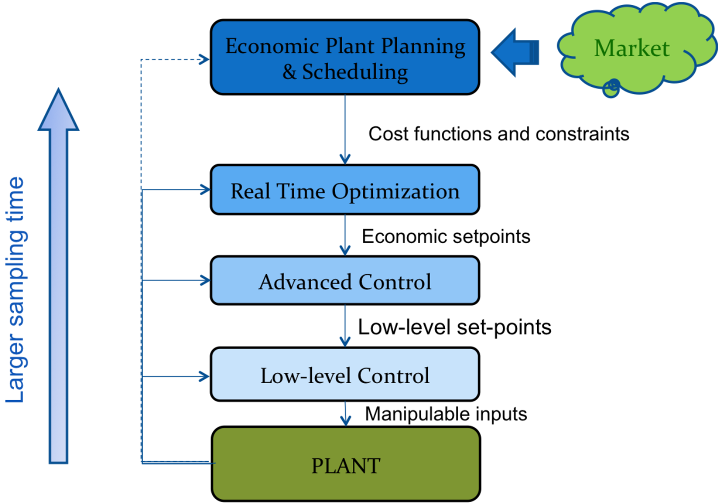
Gas and Oil value chain

A motivational example



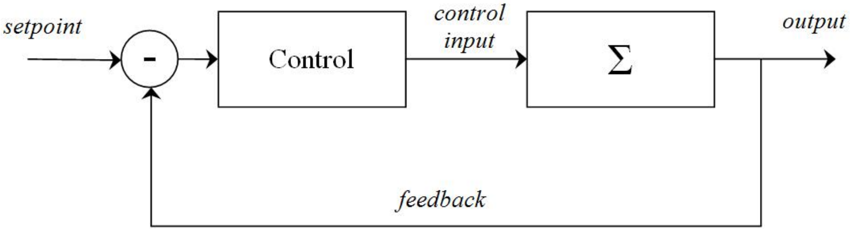
Higher Profit

Hierarchical control structure



Classic control vs. advanced control

Classic control system



Classic control vs. advanced control

	Classic control theory	Advaned control theory
Systems	Linear, time-invariant and SISO	Linear or Non-linear, time-variant or time-invariant and MIMO
Domain	frequency	time
Initial conditions	Not allowed	Allowed
Control systems design	Based on Trial and error methods, which do not allow **optimizing control techniques**	allow **optimizing control techniques**, according to arbitrary **performance indexes** and **subject to variable limitations**
System description	External: input-output polinomial description	Internal: n first order differential equation, for systems described by differential equation of order n

Optimizing control

Optimizing control concept

- A main concept regarding advanced control is that of **finding an optimal performance according to a specified objective**, subject to the system variable limits.

Optimizing control concept

- A main concept regarding advanced control is that of **finding an optimal performance according to a specified objective**, subject to the system variable limits.
- **1: Define a control objective**. What the controller should do: steering the system quickly to a given point or set; steering the system to a given point or set using the less control effort; minimizing an economic cost function in both, the path and the final point or set, etc.

Optimizing control concept

- A main concept regarding advanced control is that of **finding an optimal performance according to a specified objective**, subject to the system variable limits.
- **1: Define a control objective**. What the controller should do: steering the system quickly to a given point or set; steering the system to a given point or set using the less control effort; minimizing an economic cost function in both, the path and the final point or set, etc.
- **2: Model and Constraints**. Consider the possibilities that the system gives us to be controlled: we need a dynamic model, which includes limits for the variable as a part of the system description.

Constraints: Saturation concept

(Saturation) We will say that a variable x of a given system - constrained to be in the compact set $X \subset \mathbb{R}^{nx}$ - is saturated if it is in the boundary of X, ∂X, and this implies that the control objectives cannot be achieved.

Constraints: Saturation concept

(Saturation) We will say that a variable x of a given system - constrained to be in the compact set $X \subset \mathbb{R}^{nx}$ - is saturated if it is in the boundary of X, ∂X, and this implies that the control objectives cannot be achieved.

- If the control objective can be achieved, and to do that it necessary to keep the variable x in ∂X, then variable x is not saturated $\Longrightarrow$ it is a concept depending not only on the variable limits, but on the control objective.

Constraints: Saturation concept

(Saturation) We will say that a variable x of a given system - constrained to be in the compact set $X \subset \mathbb{R}^{nx}$ - is saturated if it is in the boundary of X, ∂X, and this implies that the control objectives cannot be achieved.

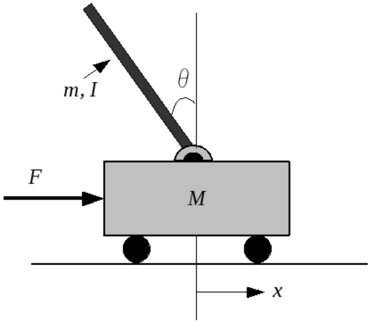
- If the control objective can be achieved, and to do that it necessary to keep the variable x in ∂X, then variable x is not saturated $\Longrightarrow$ it is a concept depending not only on the variable limits, but on the control objective.
- We could have: transient saturation (associated to the velocity of the convergence, etc), or/and stationary saturation (when the target set or point is outside the feasible set)

Constraints: Saturation concept

(Saturation) We will say that a variable x of a given system - constrained to be in the compact set $X \subset \mathbb{R}^{nx}$ - is saturated if it is in the boundary of X, ∂X, and this implies that the control objectives cannot be achieved.

- If the control objective can be achieved, and to do that it necessary to keep the variable x in ∂X, then variable x is not saturated $\Longrightarrow$ it is a concept depending not only on the variable limits, but on the control objective.
- We could have: transient saturation (associated to the velocity of the convergence, etc), or/and stationary saturation (when the target set or point is outside the feasible set)
- The concept of saturation is important in the context of Economic MPC since usually the control objective is to push the system to some limit, to maximize benefits and minimize costs.

Consider the following inverted pendulum:



Constraints: Controllability of constrained systems

State variables: position and velocity of the cart, and angular position and angular velocity of the pendulum
Controlled Output: position of the cart and angular position of the pendulum
Manipulated input: force applied to the cart. The force is subject to the following constraints:

$$\mathrm{sat}_1(u) = \begin{cases} 1 & \textit{if} \quad u > 1, \\ u & \textit{if} \quad |u| \leq 1, \\ -1 & \textit{if} \quad u < -1, \end{cases}$$

Constraints: Controllability of constrained systems

State variables: position and velocity of the cart, and angular position and angular velocity of the pendulum
Controlled Output: position of the cart and angular position of the pendulum
Manipulated input: force applied to the cart. The force is subject to the following constraints:

$$\mathrm{sat}_1(u) = \begin{cases} 1 & \textit{if} \quad u > 1, \\ u & \textit{if} \quad |u| \leq 1, \\ -1 & \textit{if} \quad u < -1, \end{cases}$$

so the constrained system can be described by

$$\begin{aligned} x(k+1) &= Ax(k) + B\mathrm{sat}_1(u(k)) \\ y(k) &= Cx(k) \end{aligned}$$

Constraints: Controllability of constrained systems

State variables: position and velocity of the cart, and angular position and angular velocity of the pendulum
Controlled Output: position of the cart and angular position of the pendulum
Manipulated input: force applied to the cart. The force is subject to the following constraints:

$$\mathrm{sat}_1(u) = \begin{cases} 1 & \text{if} \quad u > 1, \\ u & \text{if} \quad |u| \leq 1, \\ -1 & \text{if} \quad u < -1, \end{cases}$$

so the constrained system can be described by

$$\begin{aligned} x(k+1) &= Ax(k) + B\mathrm{sat}_1(u(k)) \\ y(k) &= Cx(k) \end{aligned}$$

Control Possibilities

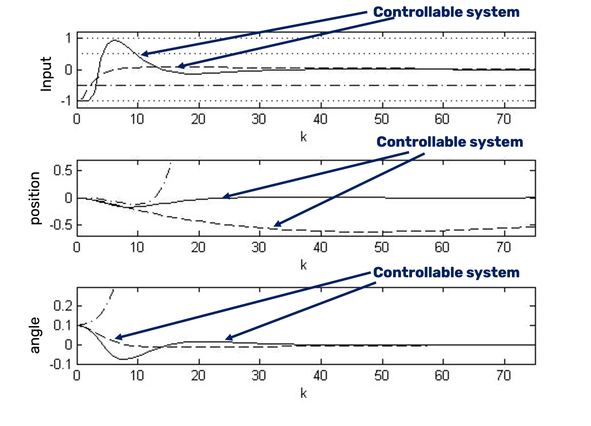
Next we will simulate a closed-loop (undefined control strategy by the moment) to see what can we do with this system.

Constraints: Controllability of constrained systems

- Assume an initial disturbance in the angular position ($\theta(0) = 0.1$) to be rejected.

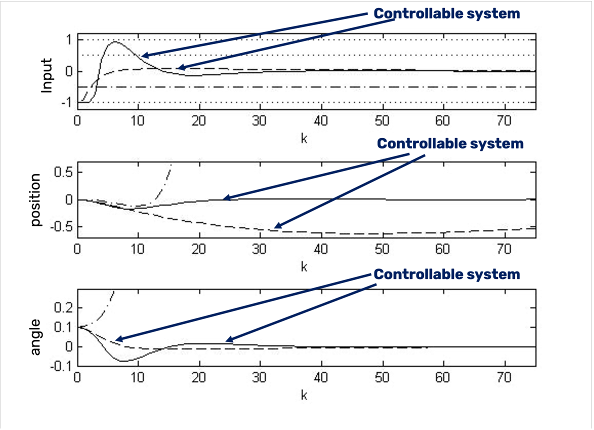
Constraints: Controllability of constrained systems

- Assume an initial disturbance in the angular position ($\theta(0) = 0.1$) to be rejected.
- **Solid line:** aggressive controller; **dashed line:** more conservative controller.



Constraints: Controllability of constrained systems

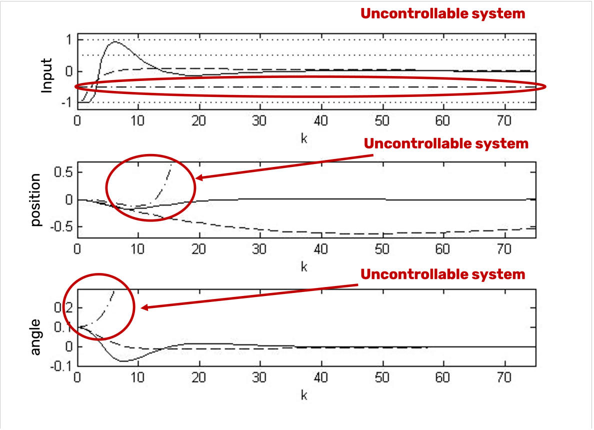
- Assume an initial disturbance in the angular position ($\theta(0) = 0.1$) to be rejected.
- **Solid line:** aggressive controller; **dashed line:** more conservative controller.



- Although the aggressive controller saturates the input at the first time instants, both controllers stabilize/control the closed-loop system.

Constraints: Controllability of constrained systems

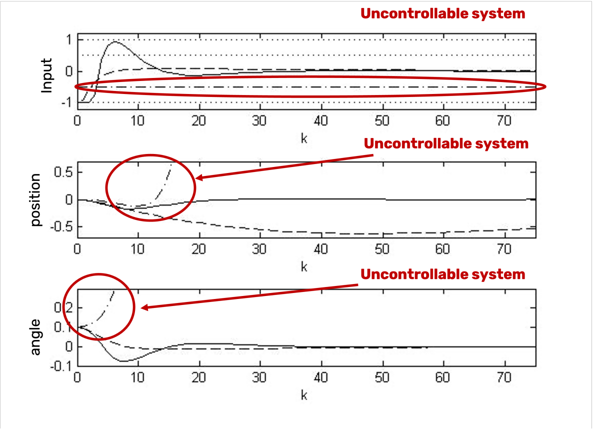
- Assume now a reduced input constraint: $\mathbb{U} = \{u : -0.5 \leq u \leq 0.5\}$



- Output: **dashed-dotted line**. Unstable closed-loop behavior (formally, the disturbance steers the system outside the maximum controllable set).

Constraints: Controllability of constrained systems

- Assume now a reduced input constraint: $\mathbb{U} = \{u : -0.5 \leq u \leq 0.5\}$



- Output: **dashed-dotted line**. Unstable closed-loop behavior (formally, the disturbance steers the system outside the maximum controllable set).
- Conclusion: **constraints affect the system controllability and stability**

How to deal with constraints

Approaches to Constrained Control:

To get the most out of a system you need to push up against limits

According to Goodgwin et al. (2005), we have the following approaches:

Approaches to Constrained Control:

To get the most out of a system you need to push up against limits

According to Goodgwin et al. (2005), we have the following approaches:

- *Cautious*: back off performance demands so constraints are not met **drawback: poor performance**.

Approaches to Constrained Control:

To get the most out of a system you need to push up against limits

According to Goodgwin et al. (2005), we have the following approaches:

- *Cautious*: back off performance demands so constraints are not met **drawback: poor performance**.
- *Serendipitous*: allow occasional constraint violation **drawback: saturation could occurs, and so unstable behavior**

Approaches to Constrained Control:

To get the most out of a system you need to push up against limits

According to Goodgwin et al. (2005), we have the following approaches:

- *Cautious*: back off performance demands so constraints are not met **drawback: poor performance**.
- *Serendipitous*: allow occasional constraint violation **drawback: saturation could occurs, and so unstable behavior**
- *Tactical*: include constraints from the beginning, in the controller design (MPC). This way, any model representation includes not only the usual model parameters, but the corresponding variable limits.

Example: linear quadratic regulator (LQR) applied to a linear system

Consider an objective function of the form:

$$V_N\left(x\left(0\right),\mathbf{u}\right)=\frac{1}{2}\sum_{k=0}^{N-1}(x(k)^TQx(k)+u(k)^TRu(k))+\frac{1}{2}x(N)^TPx(N)$$

where $\mathbf{u}$ denotes the control sequence $\{u(0),u(1),...,u(N-1)\}$ and $x(k)$ denotes the corresponding state sequence. $\mathbf{u}$ and $x(k)$ are related by the linear state equation (model):

$$x(k+1)=Ax(k)+Bu(k),\qquad k=0,1,...,N-1$$

where $x(0)$ (the initial state) is assumed to be known. The following parameters allow one to influence performance:

Example: linear quadratic regulator (LQR) applied to a linear system

Consider an objective function of the form:

$$V_N\left(x\left(0\right),\mathbf{u}\right) = \frac{1}{2}\sum_{k=0}^{N-1}(x(k)^T Qx(k) + u(k)^T Ru(k)) + \frac{1}{2}x(N)^T Px(N)$$

where $\mathbf{u}$ denotes the control sequence $\{u(0), u(1), ..., u(N-1)\}$ and $x(k)$ denotes the corresponding state sequence. $\mathbf{u}$ and $x(k)$ are related by the linear state equation (model):

$$x(k+1) = Ax(k) + Bu(k), \qquad k = 0, 1, ..., N-1$$

where $x(0)$ (the initial state) is assumed to be known. The following parameters allow one to influence performance:

- the optimization horizon N
- the state weighting matrix Q
- the control weighting matrix R
- the terminal state weighting matrix P.

For example, reducing R gives less weight on control effort, hence faster response.

Example: linear quadratic regulator (LQR) applied to a linear system

Consider the linear system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

$$\text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \qquad \text{y} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example: linear quadratic regulator (LQR) applied to a linear system

Consider the linear system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $\quad$ y $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

- This system is the zero-order hold discretisation with sampling period 1 of the double integrator: $\frac{d^2y(t)}{dt^2} = u(t)$.
- **Manipulated input:** force (acceleration and brake, depending of the sign)
- **States:** position and velocity of the mass, Initial state: $x(0) = [-6\ 0]^T$,
- **Controlled output:** position of the mass.

Example: linear quadratic regulator (LQR) applied to a linear system

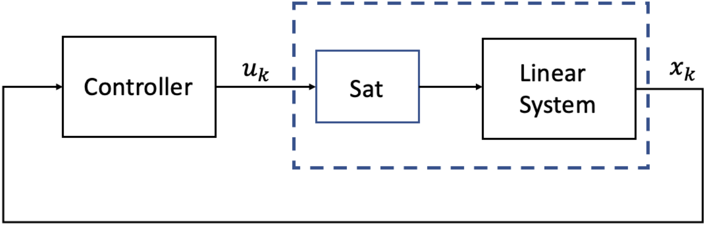
Consider the linear system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $\quad$ y $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

- This system is the zero-order hold discretisation with sampling period 1 of the double integrator: $\frac{d^2y(t)}{dt^2} = u(t)$.
- **Manipulated input:** force (acceleration and brake, depending of the sign)
- **States:** position and velocity of the mass, Initial state: $x(0) = [-6\ 0]^T$,
- **Controlled output:** position of the mass.
- The physical constraints are given by $|u(k)| \leq 1$, for all k, and they are modeled by the saturation function.

Closed-loop



Cautious Design

- $(N = \infty, P = 0)$ and weighting matrices $Q = C^T C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 20$, gives the linear state feedback law:

$$u(k) = -Kx(k) = -[0.1603 \quad 0.5662]x(k).$$

Cautious Design

- $(N = \infty, P = 0)$ and weighting matrices $Q = C^T C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 20$, gives the linear state feedback law:

$$u(k) = -Kx(k) = -[0.1603 \quad 0.5662]x(k).$$

- With this control law, the limits are not reached by the input. However, a conservative performance is obtained.

Cautious Design

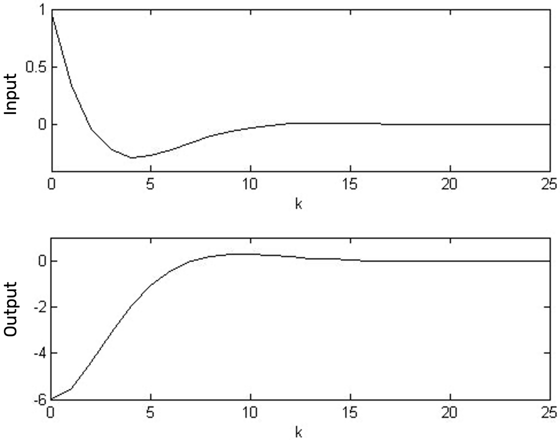


Figure: $u(k)$ and $y(k)$ for the cautious design with weights $Q = C^T C$ and $R = 20$.

Serendipitous Design

- Need a faster response? Reducing $R = 2$.

Serendipitous Design

- Need a faster response? Reducing $R = 2$.

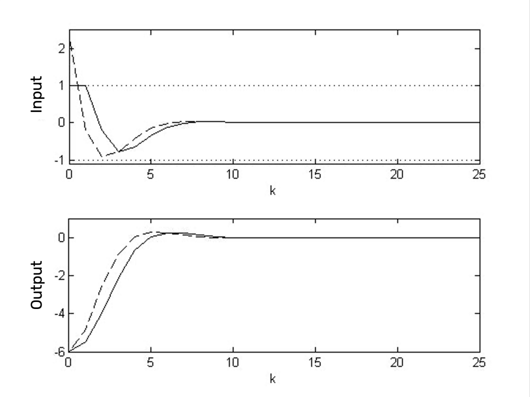


Figure: Unconstrained LQR design $u(k) = -Kx(k)$: dashed line. Serendipitous strategy, $u(k) = -\text{sat}_1(Kx(k))$: solid line.

Serendipitous Design

- Need a faster response? Reducing $R = 2$.

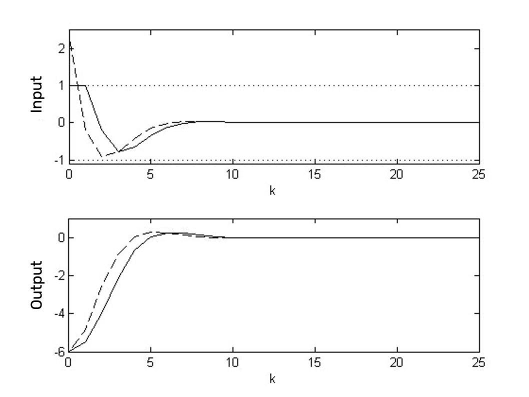


Figure: Unconstrained LQR design $u(k) = -Kx(k)$: dashed line. Serendipitous strategy, $u(k) = -\text{sat}_1(Kx(k))$: solid line.

- The input would saturate if constraints are present in the system.

Serendipitous Design

- Need a faster response? Reducing $R = 2$.

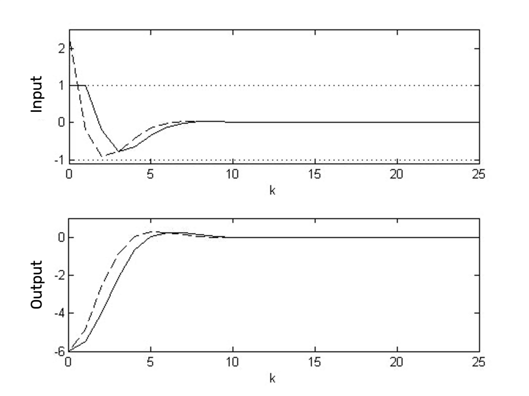


Figure: Unconstrained LQR design $u(k) = -Kx(k)$: dashed line. Serendipitous strategy, $u(k) = -\mathrm{sat}_1(Kx(k))$: solid line.

- The input would saturate if constraints are present in the system.
- Anyway, even with the saturation, the controller seems to work!

Serendipitous Design

- Let's "push our luck" further: $R = 0.1$ for an even faster response

Serendipitous Design

- Let's "push our luck" further: $R = 0.1$ for an even faster response

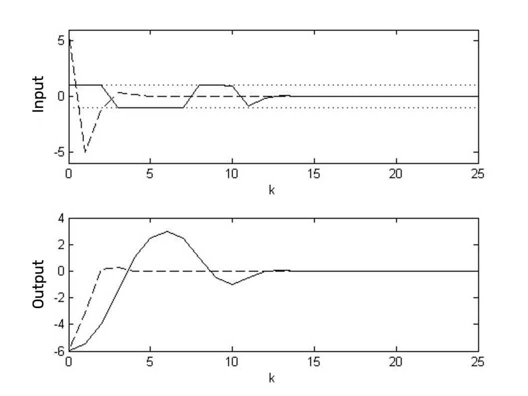


Figure: Unconstrained LQR design $u(k) = -Kx(k)$: dashed line. Serendipitous strategy, $u(k) = -\mathrm{sat}_1(Kx(k))$: solid line.

Serendipitous Design

- Let's "push our luck" further: $R = 0.1$ for an even faster response

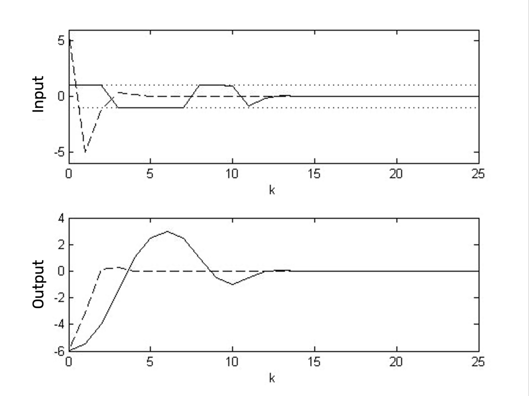


Figure: Unconstrained LQR design $u(k) = -Kx(k)$: dashed line. Serendipitous strategy, $u(k) = -\text{sat}_1(Kx(k))$: solid line.

- Unconstrained case: very fast response.

Serendipitous Design

- Let's "push our luck" further: $R = 0.1$ for an even faster response

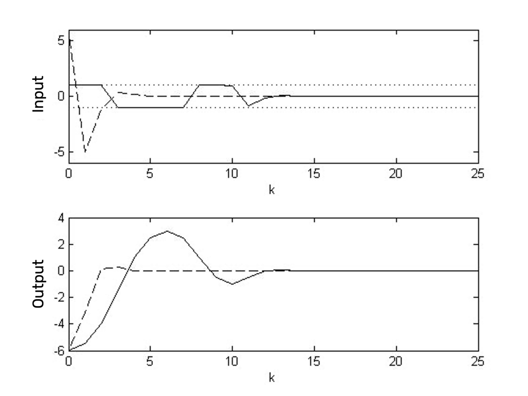
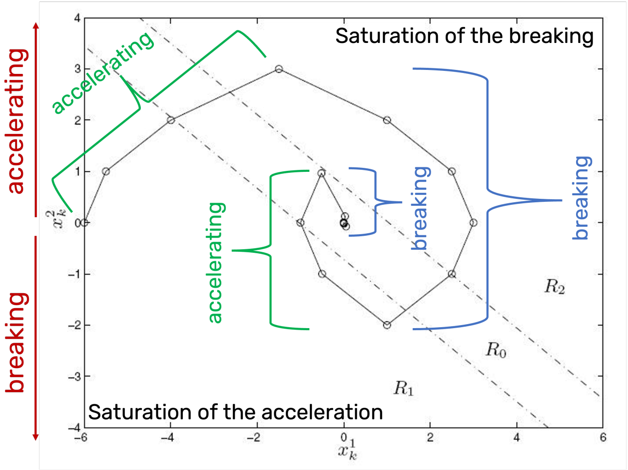


Figure: Unconstrained LQR design $u(k) = -Kx(k)$: dashed line. Serendipitous strategy, $u(k) = -\text{sat}_1(Kx(k))$: solid line.

- Saturated case (which represents reality): slower response (settling time of 12 samples).

Serendipitous Design

- The control law $u(k) = -\mathrm{sat}_1(Kx(k))$ partitions the state space into three regions in accordance with the definition of the sat function.



Serendipitous Design

- The control law $u(k) = -\text{sat}_1(Kx(k))$ partitions the state space into three regions in accordance with the definition of the sat function.
- Hence, the serendipitous strategy can be characterized as a switched control strategy in the following way:

$$u = \kappa(x) = \begin{cases} -Kx & \textit{if} \quad x \in R_0, \\ 1 & \textit{if} \quad x \in R_1, \\ -1 & \textit{if} \quad x \in R_2. \end{cases}$$

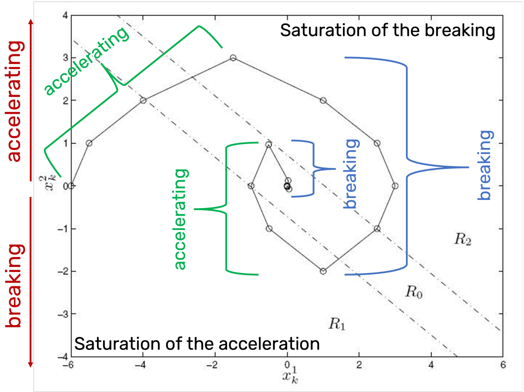
Serendipitous Design

- The control law $u(k) = -\text{sat}_1(Kx(k))$ partitions the state space into three regions in accordance with the definition of the sat function.
- Hence, the serendipitous strategy can be characterized as a switched control strategy in the following way:

$$u = \kappa(x) = \begin{cases} -Kx & \textit{if} \quad x \in R_0, \\ 1 & \textit{if} \quad x \in R_1, \\ -1 & \textit{if} \quad x \in R_2. \end{cases}$$

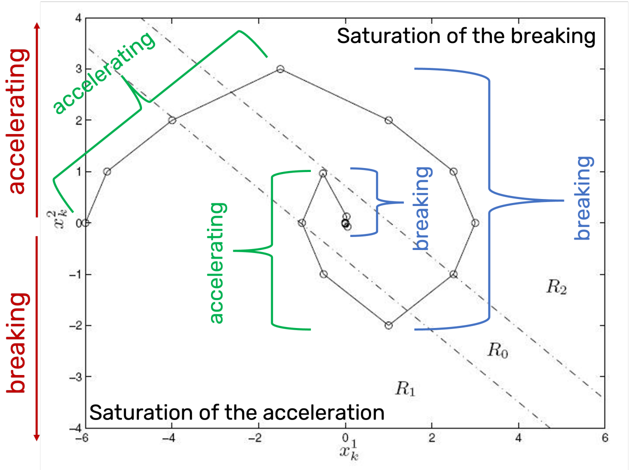
- Notice that this is simply an alternative way of describing the serendipitous strategy since for $x \in R_0$ the input actually lies between the saturation limits. The partition is shown in following figure.

Serendipitous Design



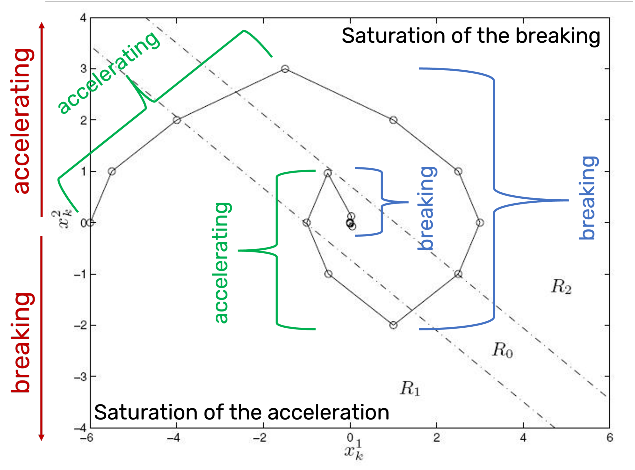
- Examination of the latter Figure suggests a heuristic argument as to why the serendipitous control law may not be performing well in this case.

Serendipitous Design



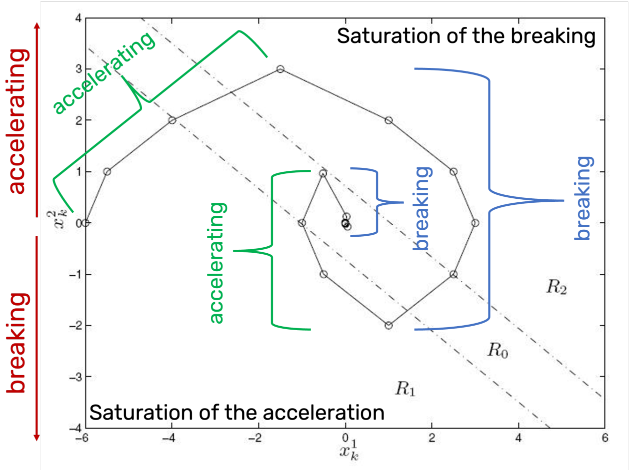
- We can think, in this example, of $x_2(k)$ as "velocity" and $x_1(k)$ as "position" of the mass.

Serendipitous Design



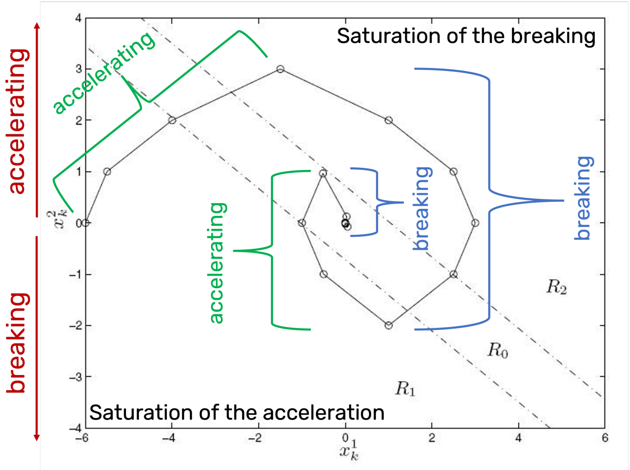
- Now, in our attempt to change the position rapidly (from -6 a 0), the velocity has been allowed to grow to a relatively high level ($+3$). **This would be fine if the braking action were unconstrained**.

Serendipitous Design



- However, our input (including braking) is limited to the range $[-1 \;\; 1]$. Hence, the available braking is inadequate to "pull the system up", and overshoot occurs.

Serendipitous Design



- Clearly, the problem with this strategy is that it does not know the constraint (variable limits), and so it cannot anticipate (predict) future saturations.

Tactical Design

- How could we remedy the problem?

Tactical Design

- How could we remedy the problem?
- A sensible idea would seem to be to try to "look ahead" and take account of future input constraints (that is, the limited braking authority available).

Tactical Design

- How could we remedy the problem?
- A sensible idea would seem to be to try to "look ahead" and take account of future input constraints (that is, the limited braking authority available).
- To test this idea, we take the objective function as a starting point.

Tactical Design

- How could we remedy the problem?
- A sensible idea would seem to be to try to "look ahead" and take account of future input constraints (that is, the limited braking authority available).
- To test this idea, we take the objective function as a starting point.
- But we now design a finite horizon constrained controller.

Tactical Design

- We use a prediction horizon $N = 2$ and minimize, at each sampling instant i and for the current state $x(i)$, the two-step objective function:

Tactical Design

- We use a prediction horizon $N = 2$ and minimize, at each sampling instant i and for the current state $x(i)$, the two-step objective function:

$$V_2\left(x\left(i\right), \mathbf{u}\right) = \frac{1}{2}\sum_{k=i}^{i+1}(x(k)^T Q x(k) + u(k)^T R u(k)) + \frac{1}{2}x(i+2)^T P x(i+2)$$

Tactical Design

- We use a prediction horizon $N = 2$ and minimize, at each sampling instant i and for the current state $x(i)$, the two-step objective function:

$$V_2\left(x\left(i\right),\mathbf{u}\right) = \frac{1}{2}\sum_{k=i}^{i+1}(x(k)^T Q x(k) + u(k)^T R u(k)) + \frac{1}{2}x(i+2)^T P x(i+2)$$

subject to the equality and inequality constraints:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ |u(k)| &\leq 1, \end{aligned}$$

for $k = i$ and $k = i + 1$.

Tactical Design

- We use a prediction horizon $N = 2$ and minimize, at each sampling instant i and for the current state $x(i)$, the two-step objective function:

$$V_2\left(x\left(i\right), \mathbf{u}\right) = \frac{1}{2}\sum_{k=i}^{i+1}(x(k)^T Q x(k) + u(k)^T R u(k)) + \frac{1}{2}x(i+2)^T P x(i+2)$$

subject to the equality and inequality constraints:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ |u(k)| &\leq 1, \end{aligned}$$

for $k = i$ and $k = i + 1$.

- In the objective function we set, as before, $Q = C^T C$, $R = 0.1$.

Tactical Design

- We use a prediction horizon $N = 2$ and minimize, at each sampling instant i and for the current state $x(i)$, the two-step objective function:

$$V_2\left(x\left(i\right),\mathbf{u}\right) = \frac{1}{2}\sum_{k=i}^{i+1}(x(k)^T Q x(k) + u(k)^T R u(k)) + \frac{1}{2}x(i+2)^T P x(i+2)$$

subject to the equality and inequality constraints:

$$\begin{array}{rcl} x(k+1) & = & Ax(k) + Bu(k), \\ |u(k)| & \leq & 1, \end{array}$$

for $k = i$ and $k = i + 1$.

- In the objective function we set, as before, $Q = C^T C$, $R = 0.1$.
- The terminal state weighting matrix P is taken to be the solution of the Riccati equation $P = A^T PA + Q - K^T (R + B^T PB)K$, where $K = (R + B^T PB)^{-1} B^T PA$ is the corresponding gain (state feedback).

Tactical Design

- As a result of minimizing the cost function subject to the constraints, we obtain an optimal fixed-horizon control sequence $\{u(i), u(i+1)\}$.

Tactical Design

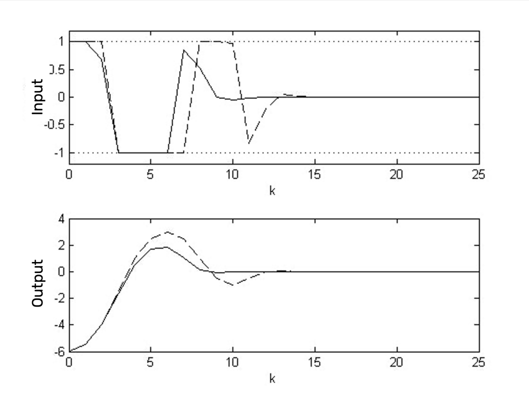
- As a result of minimizing the cost function subject to the constraints, we obtain an optimal fixed-horizon control sequence $\{u(i), u(i+1)\}$.
- We then apply the resulting value of $u(i)$ to the system. The state evolves to $x(i+1)$; the time instant is shifted from i to $i+1$, and the procedure is repeated.

Tactical Design

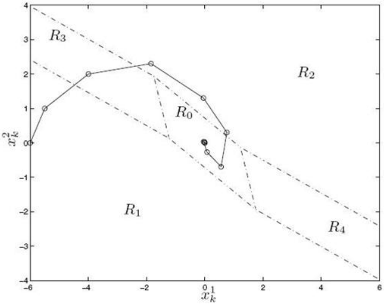
- As a result of minimizing the cost function subject to the constraints, we obtain an optimal fixed-horizon control sequence $\{u(i), u(i+1)\}$.
- We then apply the resulting value of $u(i)$ to the system. The state evolves to $x(i+1)$; the time instant is shifted from i to $i+1$, and the procedure is repeated.
- This is called **receding horizon control** [RHC] or model predictive control.

Tactical Design

- The output trajectory with constrained input now has minimal overshoot and fast response. Thus, the idea of "looking ahead" and applying the constraints in a receding horizon fashion has apparently "paid dividends."



Tactical Design



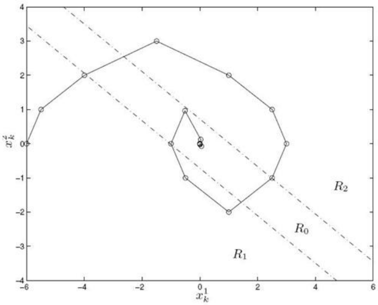


Figure: State space plot for the receding horizon tactical design and serendipitous design, respectively

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands
- However, this is at a cost

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands
- However, this is at a cost
- If we increase demands - constraints are met

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands
- However, this is at a cost
- If we increase demands - constraints are met
- Small violations not too significant

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands
- However, this is at a cost
- If we increase demands - constraints are met
- Small violations not too significant
- Large violations $\rightarrow$ poor performance

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands
- However, this is at a cost
- If we increase demands - constraints are met
- Small violations not too significant
- Large violations $\rightarrow$ poor performance
- Rethink the problem - add constraints into the design

Conclusion about Constraints

- One can often avoid constraints by lowering performance demands
- However, this is at a cost
- If we increase demands - constraints are met
- Small violations not too significant
- Large violations $\rightarrow$ poor performance
- Rethink the problem - add constraints into the design
- This leads to idea of Receding Horizon Control (**Optimizing Control**): RHC can anticipate the presence of constraints thus achieving a better control.

Model Predictive Control

Why Model Predictive Control?

- Use a dynamical model of the process (including constraints)

$$x(k+1) = f(x(k), u(k)), \quad x(k) \in \mathcal{X},\ u(k) \in \mathcal{U}$$

- to predict its future evolution by choosing the best control sequence

$$\mathbf{u} = \{u(k), u(k+1), \dots, u(k+N-1)\}$$

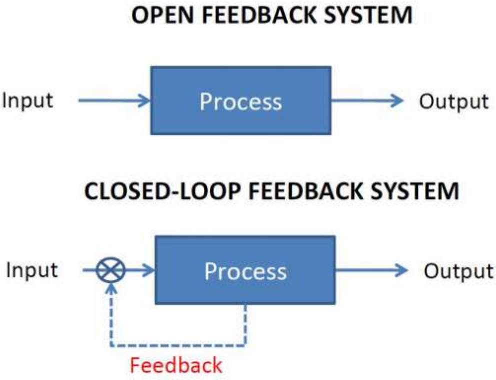
- that minimizes the performance index:

$$V_N(x; \mathbf{u}) = \sum_{j=0}^{N-1} (\|x(k+j)\|_Q^2 + \|u(k+j)\|_R^2)$$

$Q \geq 0$, $R > 0$. Prediction horizon: $N \geq 1$.

Model Predictive Control

Open-loop solution: minimize the performance index with respect to the control sequence **u**.



Receding Horizon Principle

Open-loop solution:minimize the performance index with respect to the control sequence $\mathbf{u}$.
How do we close the loop? $\rightarrow$ Receding Horizon Principle

Definition (Receding Horizon Principle)

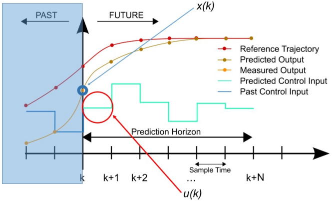
"at any time k solve the OCP over the prediction horizon [k,k+N] and apply only the first input $u^0(k)$ *of the optimal sequence* $\mathbf{u}^0(k)$*. At time k+1, move the prediction window one step ahead, and repeat the optimization over the prediction horizon [k+1,k+N+1]"*

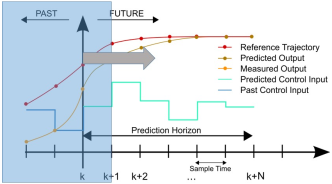
Receding Horizon

- At time k, solve an optimal control problem (OCP) over a future horizon of N steps

$$\min_{\mathbf{u}} \quad \sum_{j=0}^{N-1} (\|x(j) - x_{sp}\|_Q^2 + \|u(j) - u_{sp}\|_R^2)$$

$$\begin{aligned} s.t. \quad & x(0) = x(k) \\ & x(j+1) = f(x(j), u(j)) \\ & x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \end{aligned}$$





- Apply the first control move $u(k)$.
- At time $k+1$, get new measurement and solve the OCP. And so on...

MPC transforms open-loop control into closed-loop control.

Receding Horizon Examples

- Playing chess.

- Driving.

MPC with linear systems

- Dynamical model of the form

$$x(k+1) = Ax(k) + Bu(k), \quad x(k) \in \mathcal{X},\, u(k) \in \mathcal{U}$$

MPC with linear systems

- Dynamical model of the form

$$x(k+1) = Ax(k) + Bu(k), \quad x(k) \in \mathcal{X},\, u(k) \in \mathcal{U}$$

- then the OCP

$$\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{j=0}^{N-1} (\|x(j) - x_{sp}\|_Q^2 + \|u(j) - u_{sp}\|_R^2) \\
s.t. \quad & x(0) = x(k), \quad x(j+1) = Ax(j) + Bu(j) \\
& x(j) \in \mathcal{X},\, u(j) \in \mathcal{U},\, j \in \mathbb{I}_{[0,N-1]}
\end{aligned}$$

is a convex Quadratic Programming (QP) problem (Boyd & Vandenberghe, 2006)

$$\begin{aligned}
\min_{\mathbf{u}} \quad & \frac{1}{2}\mathbf{u}'H\mathbf{u} + f'\mathbf{u} + r \\
s.t. \quad & G\mathbf{u} \leq W
\end{aligned}$$

Model Predictive Control

Widely used in the process industries Camacho & Bordons (2004)

- Usually in its "original" Dynamic Matrix Control (DMC) formulation (AspenTech, Honeywll, ABB).

Model Predictive Control

Widely used in the process industries Camacho & Bordons (2004)

- Usually in its "original" Dynamic Matrix Control (DMC) formulation (AspenTech, Honeywll, ABB).

Model Predictive Control

Widely used in the process industries Camacho & Bordons (2004)

- Usually in its "original" Dynamic Matrix Control (DMC) formulation (AspenTech, Honeywll, ABB).

Stability theory very mature(Mayne et al., 2000; Rawlings & Mayne, 2009):

Model Predictive Control

Widely used in the process industries Camacho & Bordons (2004)

- Usually in its "original" Dynamic Matrix Control (DMC) formulation (AspenTech, Honeywll, ABB).

Stability theory very mature(Mayne et al., 2000; Rawlings & Mayne, 2009):

- Terminal equality constraint.
- Terminal cost.
- Terminal inequality constraint.
- Terminal cost + Terminal inequality constraint.
- Robust stability: ISS as a general framework (Limon et al., 2009).

Model Predictive Control

Widely used in the process industries Camacho & Bordons (2004)

- Usually in its "original" Dynamic Matrix Control (DMC) formulation (AspenTech, Honeywll, ABB).

Stability theory very mature(Mayne et al., 2000; Rawlings & Mayne, 2009):

- Terminal equality constraint.
- Terminal cost.
- Terminal inequality constraint.
- Terminal cost + Terminal inequality constraint.
- Robust stability: ISS as a general framework (Limon et al., 2009).

Based on this, the optimal performance index can be considered as a

Lyapunov function

MPC Stability

Some Definitions.

Linear vector space

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

Linear vector space

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)

Linear vector space

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)

Linear vector space

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)
3. there exists an element $0 \in \mathcal{X}$ s. t. $0 + x = x + 0 = x, \quad \forall x \in \mathcal{X}$ (existence of the **neutral** element w.r.t. the sum)

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)
3. there exists an element $0 \in \mathcal{X}$ s. t. $0 + x = x + 0 = x, \quad \forall x \in \mathcal{X}$ (existence of the **neutral** element w.r.t. the sum)
4. for all $x \in \mathcal{X}$ there exists an element $-x \in \mathcal{X}$ s. t. $x + (-x) = 0$ (existence of the **inverse** element w.r.t. the sum)

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)
3. there exists an element $0 \in \mathcal{X}$ s. t. $0 + x = x + 0 = x, \quad \forall x \in \mathcal{X}$ (existence of the **neutral** element w.r.t. the sum)
4. for all $x \in \mathcal{X}$ there exists an element $-x \in \mathcal{X}$ s. t. $x + (-x) = 0$ (existence of the **inverse** element w.r.t. the sum)
5. for all $r_1, r_2 \in \mathbb{R}$ ($c_1, c_2 \in \mathbb{C}$), and each $x \in \mathcal{X}$, there exists an element $r_1 \cdot (r_2 \cdot x) = (r_1 r_2) \cdot x$ ($c_1 \cdot (c_2 \cdot x) = (c_1 c_2) \cdot x$)

Linear vector space

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)
3. there exists an element $0 \in \mathcal{X}$ s. t. $0 + x = x + 0 = x, \quad \forall x \in \mathcal{X}$ (existence of the **neutral** element w.r.t. the sum)
4. for all $x \in \mathcal{X}$ there exists an element $-x \in \mathcal{X}$ s. t. $x + (-x) = 0$ (existence of the **inverse** element w.r.t. the sum)
5. for all $r_1, r_2 \in \mathbb{R}$ ($c_1, c_2 \in \mathbb{C}$), and each $x \in \mathcal{X}$, there exists an element $r_1 \cdot (r_2 \cdot x) = (r_1 r_2) \cdot x$ $(c_1 \cdot (c_2 \cdot x) = (c_1 c_2) \cdot x)$
6. for all $r \in \mathbb{R}$ ($c \in \mathbb{C}$), and each $x, y \in \mathcal{X}$, there exists an element $r \cdot (x + y) = r \cdot x + r \cdot y$ $(c \cdot (x + y) = c \cdot x + c \cdot y)$

Linear vector space

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)
3. there exists an element $0 \in \mathcal{X}$ s. t. $0 + x = x + 0 = x, \quad \forall x \in \mathcal{X}$ (existence of the **neutral** element w.r.t. the sum)
4. for all $x \in \mathcal{X}$ there exists an element $-x \in \mathcal{X}$ s. t. $x + (-x) = 0$ (existence of the **inverse** element w.r.t. the sum)
5. for all $r_1, r_2 \in \mathbb{R}$ ($c_1, c_2 \in \mathbb{C}$), and each $x \in \mathcal{X}$, there exists an element $r_1 \cdot (r_2 \cdot x) = (r_1 r_2) \cdot x$ $(c_1 \cdot (c_2 \cdot x) = (c_1 c_2) \cdot x)$
6. for all $r \in \mathbb{R}$ ($c \in \mathbb{C}$), and each $x, y \in \mathcal{X}$, there exists an element $r \cdot (x + y) = r \cdot x + r \cdot y$ $(c \cdot (x + y) = c \cdot x + c \cdot y)$
7. for all $r_1, r_2 \in \mathbb{R}$ ($c_1, c_2 \in \mathbb{C}$), and each $x \in \mathcal{X}$, there exists an element $(r_1 + r_2) \cdot x = r_1 \cdot x + r_2 \cdot x$ $((c_1 + c_2) \cdot x = c_1 \cdot x + c_2 \cdot x)$

Linear vector space

A linear vector space, or simply vector space (real or complex) is a set $\mathcal{X}$ where two operations are defined $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, called sum, and $\bullet : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ (or $\bullet : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$), called scalar product, such that the following axioms are verified:

1. $x + y = y + x, \quad \forall x, y \in \mathcal{X}$ (commutativity of the sum)
2. $x + (y + z) = (y + x) + z, \quad \forall x, y, z \in \mathcal{X}$ (associativity of the sum)
3. there exists an element $0 \in \mathcal{X}$ s. t. $0 + x = x + 0 = x, \quad \forall x \in \mathcal{X}$ (existence of the **neutral** element w.r.t. the sum)
4. for all $x \in \mathcal{X}$ there exists an element $-x \in \mathcal{X}$ s. t. $x + (-x) = 0$ (existence of the **inverse** element w.r.t. the sum)
5. for all $r_1, r_2 \in \mathbb{R}$ ($c_1, c_2 \in \mathbb{C}$), and each $x \in \mathcal{X}$, there exists an element $r_1 \cdot (r_2 \cdot x) = (r_1 r_2) \cdot x$ ($c_1 \cdot (c_2 \cdot x) = (c_1 c_2) \cdot x$)
6. for all $r \in \mathbb{R}$ ($c \in \mathbb{C}$), and each $x, y \in \mathcal{X}$, there exists an element $r \cdot (x + y) = r \cdot x + r \cdot y$ ($c \cdot (x + y) = c \cdot x + c \cdot y$)
7. for all $r_1, r_2 \in \mathbb{R}$ ($c_1, c_2 \in \mathbb{C}$), and each $x \in \mathcal{X}$, there exists an element $(r_1 + r_2) \cdot x = r_1 \cdot x + r_2 \cdot x$ ($(c_1 + c_2) \cdot x = c_1 \cdot x + c_2 \cdot x$)
8. for all $x \in \mathcal{X}$, $1 \cdot x = x$.

Normed vector space

The concept of distance between elements, is fundamental in order to add concept that we are interested in, such as *convergence* and *continuity*. If we add such a concept, we can define the so called **normed vector spaces**.

Normed vector space

The concept of distance between elements, is fundamental in order to add concept that we are interested in, such as *convergence* and *continuity*. If we add such a concept, we can define the so called **normed vector spaces**.

Normed vector space

A normed vector space is a couple $\mathcal{X}, \|\cdot\|$, where $\mathcal{X}$ is a vector space and $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a real function, called norm, such that:

Normed vector space

The concept of distance between elements, is fundamental in order to add concept that we are interested in, such as *convergence* and *continuity*. If we add such a concept, we can define the so called **normed vector spaces**.

Normed vector space

A normed vector space is a couple $\mathcal{X}, \|\cdot\|$, where $\mathcal{X}$ is a vector space and $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a real function, called norm, such that:

1. $\|x\| \geq 0$, for all $x \in \mathcal{X}$; $\|x\| = 0$ iff $x = 0$

Normed vector space

The concept of distance between elements, is fundamental in order to add concept that we are interested in, such as *convergence* and *continuity*. If we add such a concept, we can define the so called **normed vector spaces**.

Normed vector space

A normed vector space is a couple $\mathcal{X}, \|\cdot\|$, where $\mathcal{X}$ is a vector space and $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a real function, called norm, such that:

1. $\|x\| \geq 0$, for all $x \in \mathcal{X}$; $\|x\| = 0$ iff $x = 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$, for all $x \in \mathcal{X}$, and all α

Normed vector space

The concept of distance between elements, is fundamental in order to add concept that we are interested in, such as *convergence* and *continuity*. If we add such a concept, we can define the so called **normed vector spaces**.

Normed vector space

A normed vector space is a couple $\mathcal{X}, \|\cdot\|$, where $\mathcal{X}$ is a vector space and $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a real function, called norm, such that:

1. $\|x\| \geq 0$, for all $x \in \mathcal{X}$; $\|x\| = 0$ iff $x = 0$
2. $\|\alpha x\| = |\alpha|\,\|x\|$, for all $x \in \mathcal{X}$, and all α
3. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in \mathcal{X}$ (triangle inequality).

Normed vector space

The concept of distance between elements, is fundamental in order to add concept that we are interested in, such as *convergence* and *continuity*. If we add such a concept, we can define the so called **normed vector spaces**.

Normed vector space

A normed vector space is a couple $\mathcal{X}, ||\cdot||$, where $\mathcal{X}$ is a vector space and $||\cdot|| : \mathcal{X} \to \mathbb{R}$ is a real function, called norm, such that:

1. $||x|| \geq 0$, for all $x \in \mathcal{X}$; $||x|| = 0$ iff $x = 0$
2. $||\alpha x|| = |\alpha| \, ||x||$, for all $x \in \mathcal{X}$, and all α
3. $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in \mathcal{X}$ (triangle inequality).

Remark

Note that the concept of norm in a normed vector space is a generalization of the concept of distance in $\mathbb{R}^2$ $\mathbb{R}^3$. *Then,* $||x - y||$ *can be seen as the distance between the two vectors or elements* x *and* y. *This concept allows us to define the notions of convergence and proximity in a vector space.*

Examples of norm oeprators

In general a p-norm in $\mathbb{R}^n$ is defined as:

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + ... + |x_n|^p}$$

Examples of norm oeprators

In general a p-norm in $\mathbb{R}^n$ is defined as:

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \ldots + |x_n|^p}$$

- 1-norm

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

Examples of norm oeprators

In general a p-norm in $\mathbb{R}^n$ is defined as:

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}$$

- 1-norm

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

- 2-norm (euclidean norm)

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

Examples of norm oeprators

In general a p-norm in $\mathbb{R}^n$ is defined as:

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \ldots + |x_n|^p}$$

- 1-norm

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

- 2-norm (euclidean norm)

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$$

- ∞-norm

$$\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|) = \max_{i \in \{1,2,\ldots,n\}} |x_i|$$

Examples of norm oeprators

In general a p-norm in $\mathbb{R}^n$ is defined as:

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \ldots + |x_n|^p}$$

- 1-norm

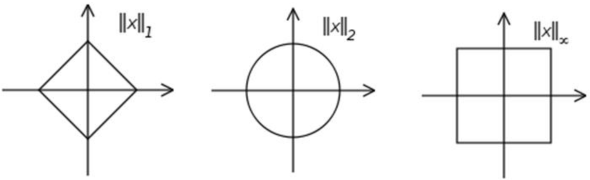
$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

- 2-norm (euclidean norm)

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$$

- ∞-norm

$$\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|) = \max_{i \in \{1,2,\ldots,n\}} |x_i|$$



Convergence y continuity

We can now define the concepts of *convergence* and *continuity* in a vector space.

Convergence y continuity

We can now define the concepts of *convergence* and *continuity* in a vector space.

Convergence

Let $\{x_n\}_1^\infty$ be a sequence of elements belonging to a normed vector space $(\mathcal{X}, \|\cdot\|)$. We say that such a sequence *converges* to the element $x_0 \in \mathcal{X}$ if $\|x_n - x_0\| \to 0$ for $n \to \infty$.

Convergence y continuity

We can now define the concepts of *convergence* and *continuity* in a vector space.

Convergence

Let $\{x_n\}_1^\infty$ be a sequence of elements belonging to a normed vector space $(\mathcal{X}, \|\cdot\|)$. We say that such a sequence *converges* to the element $x_0 \in \mathcal{X}$ if $\|x_n - x_0\| \to 0$ for $n \to \infty$.
That is, $\{x_n\}_1^\infty$ *converges* to x_0 if, for all $\epsilon > 0$, there exists and integer $N(\epsilon)$ such that

$$\|x_n - x_0\| < \epsilon, \quad \text{for all } \; n \geq N(\epsilon).$$

Convergence y continuity

We can now define the concepts of *convergence* and *continuity* in a vector space.

Convergence

Let $\{x_n\}_1^\infty$ be a sequence of elements belonging to a normed vector space $(\mathcal{X}, \|\cdot\|)$. We say that such a sequence *converges* to the element $x_0 \in \mathcal{X}$ if $\|x_n - x_0\| \to 0$ for $n \to \infty$.
That is, $\{x_n\}_1^\infty$ *converges* to x_0 if, for all $\epsilon > 0$, there exists and integer $N(\epsilon)$ such that

$$\|x_n - x_0\| < \epsilon, \quad \text{for all } \; n \geq N(\epsilon).$$

Remark

We can give different interpretations to the previous definition:

- $\{x_n\}_1^\infty \to x_0$ *iff the sequence* $\{\|x_n - x_0\|\}_1^\infty \to 0$.

Convergence y continuity

We can now define the concepts of *convergence* and *continuity* in a vector space.

Convergence

Let $\{x_n\}_1^\infty$ be a sequence of elements belonging to a normed vector space $(\mathcal{X}, \|\cdot\|)$. We say that such a sequence *converges* to the element $x_0 \in \mathcal{X}$ if $\|x_n - x_0\| \to 0$ for $n \to \infty$.
That is, $\{x_n\}_1^\infty$ *converges* to x_0 if, for all $\epsilon > 0$, there exists and integer $N(\epsilon)$ such that

$$\|x_n - x_0\| < \epsilon, \quad \text{for all } \ n \geq N(\epsilon).$$

Remark

We can give different interpretations to the previous definition:

- $\{x_n\}_1^\infty \to x_0$ *iff the sequence* $\{\|x_n - x_0\|\}_1^\infty \to 0$.
- *Let* $B(x_0, \epsilon) = \{x \in \mathcal{X} : \|x - x_0\| < \epsilon\}$*; then* $\{x_n\}_1^\infty \to x_0$ *iff for all* $\epsilon > 0$, $B(x_0, \epsilon)$ *contains all -but a finite number of - the elements of the sequence* $\{x_n\}_1^\infty$.

Convergence y continuity

Continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *continuous in* $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$ there exists a $\delta(\epsilon, x_0) > 0$ such that

$$\text{if} \quad \|x_0 - x\|_{\mathcal{X}} < \delta(\epsilon, x_0), \quad \text{then} \quad \|f(x_0) - f(x)\|_{\mathcal{Y}} < \epsilon.$$

Convergence y continuity

Continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *continuous in* $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$ there exists a $\delta(\epsilon, x_0) > 0$ such that

$$\text{if} \quad \|x_0 - x\|_{\mathcal{X}} < \delta(\epsilon, x_0), \quad \text{then} \quad \|f(x_0) - f(x)\|_{\mathcal{Y}} < \epsilon.$$

f is *continuous* if it is continuous in all $x \in \mathcal{X}$

Continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *continuous in* $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$ there exists a $\delta(\epsilon, x_0) > 0$ such that

$$\text{if } \ \|x_0 - x\|_{\mathcal{X}} < \delta(\epsilon, x_0), \ \text{ then } \ \|f(x_0) - f(x)\|_{\mathcal{Y}} < \epsilon.$$

f is *continuous* if it is continuous in all $x \in \mathcal{X}$
f is *uniformly continuous* if it is continuous and for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\text{if } \ \|x_0 - x\|_{\mathcal{X}} < \delta(\epsilon), \ \text{ then } \ \|f(x_0) - f(x)\|_{\mathcal{Y}} < \epsilon.$$

Convergence y continuity

Continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *continuous in* $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$ there exists a $\delta(\epsilon, x_0) > 0$ such that

$$\text{if} \quad \|x_0 - x\|_{\mathcal{X}} < \delta(\epsilon, x_0), \quad \text{then} \quad \|f(x_0) - f(x)\|_{\mathcal{Y}} < \epsilon.$$

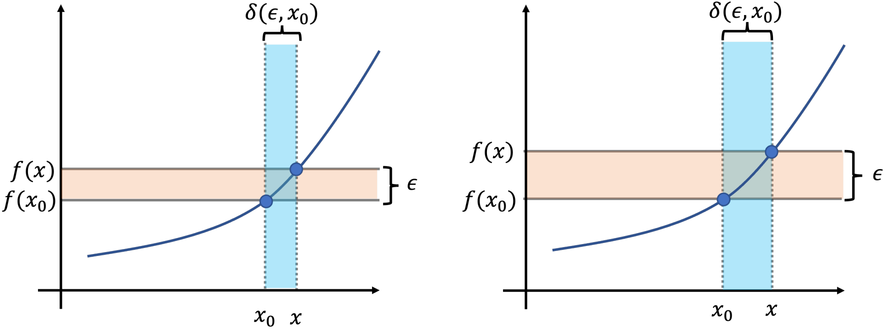
f is *continuous* if it is continuous in all $x \in \mathcal{X}$
f is *uniformly continuous* if it is continuous and for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\text{if} \quad \|x_0 - x\|_{\mathcal{X}} < \delta(\epsilon), \quad \text{then} \quad \|f(x_0) - f(x)\|_{\mathcal{Y}} < \epsilon.$$

Remark

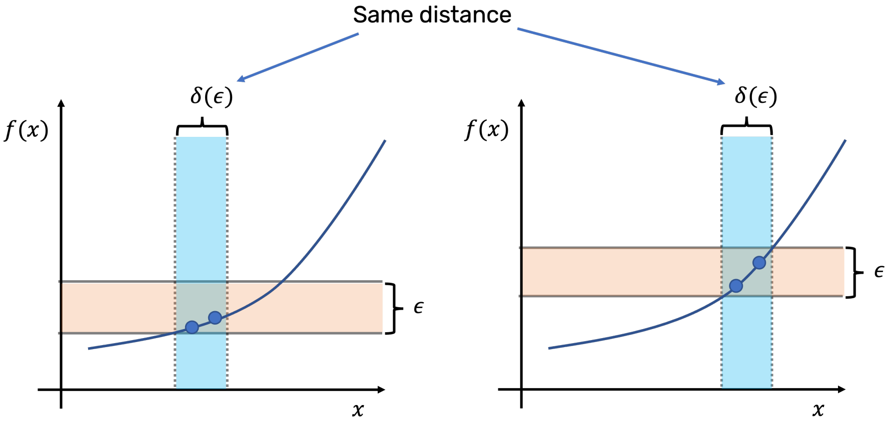
The difference between continuity and uniform continuity lies in the fact that in the second definition δ only depends on ϵ and not on x.

Continuous function



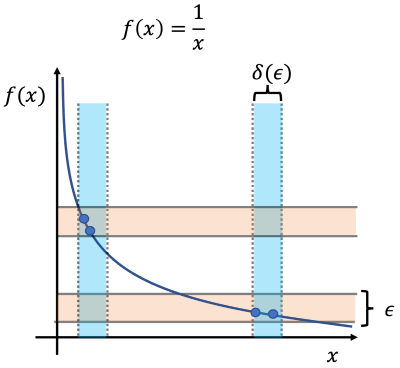
We want the distance between two points to be smaller then the error between images.

Uniformly continuous function



Given the distance between two images, the distance between two points is always smaller than the error between images, for all points.

Counter example: continuous, but not uniformly, functi



For a given ϵ between images, there not exists a δ smaller than the error, for all points.

The slope tends to infinite if x tends to 0.

Function $f(x) = \frac{1}{x}$ is not uniformly continuous in $[0, \infty)$. It is however uniformly continuous in any interval $[a, \infty)$, with $a > 0$.

Convergence y continuity

Very close to the previous concepts, there's the concept of Lipschitz continuity.

Lipschitz continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let f be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *Lipschitz continuous in* $\mathcal{S} \subseteq \mathcal{X}$ if there exists a real constant $L \geq 0$ such that

$$\|f(x_1) - f(x_2)\|_{\mathcal{Y}} \leq L\,\|x_1 - x_2\|_{\mathcal{X}} \quad \text{for all } \; x_1, x_2 \in \mathcal{S}.$$

Convergence y continuity

Very close to the previous concepts, there's the concept of Lipschitz continuity.

Lipschitz continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let f be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *Lipschitz continuous in* $\mathcal{S} \subseteq \mathcal{X}$ if there exists a real constant $L \geq 0$ such that

$$\|f(x_1) - f(x_2)\|_{\mathcal{Y}} \leq L \|x_1 - x_2\|_{\mathcal{X}} \quad \text{for all } \; x_1, x_2 \in \mathcal{S}.$$

Theorem

Lipschitz continuous function are uniformly continuous.

Convergence y continuity

Very close to the previous concepts, there's the concept of Lipschitz continuity.

Lipschitz continuity

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed vector spaces. Let f be a *mapping* from $\mathcal{X}$ to $\mathcal{Y}$. We state that f is *Lipschitz continuous in* $\mathcal{S} \subseteq \mathcal{X}$ if there exists a real constant $L \geq 0$ such that

$$\|f(x_1) - f(x_2)\|_{\mathcal{Y}} \leq L\,\|x_1 - x_2\|_{\mathcal{X}} \quad \text{for all} \;\; x_1, x_2 \in \mathcal{S}.$$

Theorem

Lipschitz continuous function are uniformly continuous.

Remark

Let's suppose that $f : \mathbb{R} \to \mathbb{R}$*. Due to the mean value theorem, if* f *is differentiable in* $[x_1, x_2]$*, then*

$$|f(x_1) - f(x_2)| \leq f'(c)\,|x_1 - x_2| \quad \textit{for some} \;\; c \in [x_1, x_2].$$

Therefore, if x_1 *y* x_2 *belong to an interval* $\mathcal{S}$ *and* $f'(c) \leq L$*, for all* $c \in \mathcal{S}$*, then it follows that* f *is Lipschitz continuous.*

Compact sets

Closed set

A set $S \in \mathcal{X}$ is *closed* iff all converging sequences, with elements in S, have limit in S.

Compact sets

Closed set

A set $S \in \mathcal{X}$ is *closed* iff all converging sequences, with elements in S, have limit in S.

Remark

Different definitions (i) A closed set can be defined as a set which contains all its limit points.
(ii) A set is closed if its complement is an open set.
(iii) Roughly speaking, a set is closed if it contains its boundary.

Compact sets

Closed set

A set $S \in \mathcal{X}$ is *closed* iff all converging sequences, with elements in S, have limit in S.

Remark

Different definitions (i) A closed set can be defined as a set which contains all its limit points.
(ii) A set is closed if its complement is an open set.
(iii) Roughly speaking, a set is closed if it contains its boundary.

Bounded set

A set $S \in \mathcal{X}$ is *bounded* if it is in some sense of finite size.

Compact sets

Closed set

A set $S \in \mathcal{X}$ is *closed* iff all converging sequences, with elements in S, have limit in S.

Remark

Different definitions (i) A closed set can be defined as a set which contains all its limit points.
(ii) A set is closed if its complement is an open set.
(iii) Roughly speaking, a set is closed if it contains its boundary.

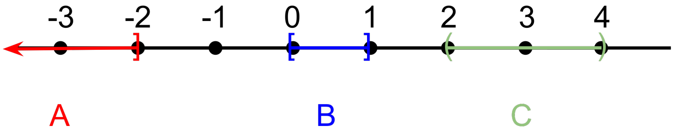
Bounded set

A set $S \in \mathcal{X}$ is *bounded* if it is in some sense of finite size.

Compact set

A set $S \in \mathcal{X}$ is *compact* if it is closed and bounded.

Compact sets



- The interval $A = (-\infty, -2]$ is not compact because it is not bounded.
- The interval $C = (2, 4)$ is not compact because it is not closed.
- The interval $B = [0, 1]$ is compact because it is both bounded and closed.

Cauchy sequence

A sequence $\{x_n\}_1^\infty$ in a normed vector space $(\mathcal{X}, \|\cdot\|)$ converges to x_0 if $\|x_n - x_0\|$ tends to zero when $n \to \infty$.

However, in many cases the limit point of a certain sequence is unknown. This is the case for instance, of iterative solutions to differential equations.

We then need a different way to characterize a sequence, which does not depend on the (unknown) limit point if such a sequence. We introduce the concept of Cauchy sequence.

Cauchy sequence

A sequence $\{x_n\}_1^\infty$ in a normed vector space $(\mathcal{X}, \|\cdot\|)$ converges to x_0 if $\|x_n - x_0\|$ tends to zero when $n \to \infty$.

However, in many cases the limit point of a certain sequence is unknown. This is the case for instance, of iterative solutions to differential equations.

We then need a different way to characterize a sequence, which does not depend on the (unknown) limit point if such a sequence. We introduce the concept of Cauchy sequence.

Cauchy sequence

Let $\{x_n\}_1^\infty$ be a sequence of elements belonging to a normed vector space $(\mathcal{X}, \|\cdot\|)$. We say that such a sequence is a *Cauchy sequence* if, for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\|x_n - x_m\| < \epsilon, \quad \text{for all } \ n, m \geq N(\epsilon).$$

Cauchy sequence

A sequence $\{x_n\}_1^\infty$ in a normed vector space $(\mathcal{X}, \|\cdot\|)$ converges to x_0 if $\|x_n - x_0\|$ tends to zero when $n \to \infty$.

However, in many cases the limit point of a certain sequence is unknown. This is the case for instance, of iterative solutions to differential equations.

We then need a different way to characterize a sequence, which does not depend on the (unknown) limit point if such a sequence. We introduce the concept of Cauchy sequence.

Cauchy sequence

Let $\{x_n\}_1^\infty$ be a sequence of elements belonging to a normed vector space $(\mathcal{X}, \|\cdot\|)$. We say that such a sequence is a *Cauchy sequence* if, for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\|x_n - x_m\| < \epsilon, \quad \text{for all } \; n, m \geq N(\epsilon).$$

Remark

Based on the above definition, a sequence $\{x_n\}_1^\infty$ is convergent if its terms x_n get arbitrarily closer to a fixed element x_0. On the other hand, it will be a Cauchy sequence if its terms get arbitrarily closer to each other, when $n \to \infty$.

Cauchy sequence

Proposition

Any convergent sequence in a norm vector space is a Cauchy sequence.

Cauchy sequence

Proposition

Any convergent sequence in a norm vector space is a Cauchy sequence.

Remark

Although any convergent sequence in a norm vector space is a Cauchy sequence, the inverse is in general not true: that is, not all Cauchy sequences are convergent.

Cauchy sequence

Proposition

Any convergent sequence in a norm vector space is a Cauchy sequence.

Remark

Although any convergent sequence in a norm vector space is a Cauchy sequence, the inverse is in general not true: that is, not all Cauchy sequences are convergent.

However, some normed vector spaces enjoy the special property that any Cauchy sequence in a norm vector space is a convergent sequence.

Cauchy sequence

Proposition

Any convergent sequence in a norm vector space is a Cauchy sequence.

Remark

Although any convergent sequence in a norm vector space is a Cauchy sequence, the inverse is in general not true: that is, not all Cauchy sequences are convergent.

However, some normed vector spaces enjoy the special property that any Cauchy sequence in a norm vector space is a convergent sequence.

Banach space

A normed vector space $(\mathcal{X}, \|\cdot\|)$ is *complete*, or a *Banach* space, if any Cauchy sequence **converges**, and converges to a un element of $\mathcal{X}$.

The sets $\mathbb{R}^n$ y $\mathbb{C}^n$, for all $n \in \mathbb{N}$, are Banach spaces.

Constrained Nonlinear System definitions

- autonomous system (S), $x^+ = f(x)$, $x \in \mathcal{X}$
- autonomous linear system (SLin), $x^+ = Ax$, $x \in \mathcal{X}$,

Constrained Nonlinear System definitions

- autonomous system (S), $x^+ = f(x)$, $x \in \mathcal{X}$
- autonomous linear system (SLin), $x^+ = Ax$, $x \in \mathcal{X}$,
- controlled system (CS), $x^+ = f(x,u)$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- controlled linear system (CSLin), $x^+ = Ax + Bu$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,

Constrained Nonlinear System definitions

- autonomous system (S), $x^+ = f(x)$, $x \in \mathcal{X}$
- autonomous linear system (SLin), $x^+ = Ax$, $x \in \mathcal{X}$,
- controlled system (CS), $x^+ = f(x,u)$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- controlled linear system (CSLin), $x^+ = Ax + Bu$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- Closed-loop system (CLS), $x^+ = f(x, \kappa(x))$, $(x, \kappa(x)) \in \mathcal{Z}$
- Closed-loop system (CLSLin), $x^+ = Ax + B\kappa(x)$, $(x, \kappa(x)) \in \mathcal{Z}$

Constrained Nonlinear System definitions

- autonomous system (S), $x^+ = f(x)$, $x \in \mathcal{X}$
- autonomous linear system (SLin), $x^+ = Ax$, $x \in \mathcal{X}$,
- controlled system (CS), $x^+ = f(x,u)$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- controlled linear system (CSLin), $x^+ = Ax + Bu$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- Closed-loop system (CLS), $x^+ = f(x, \kappa(x))$, $(x, \kappa(x)) \in \mathcal{Z}$
- Closed-loop system (CLSLin), $x^+ = Ax + B\kappa(x)$, $(x, \kappa(x)) \in \mathcal{Z}$
- $\mathcal{X} \in \mathbb{R}^n$ closed and convex, and $\mathcal{U} \in \mathbb{R}^m$ and compact (closed and bounded) and convex.

Constrained Nonlinear System definitions

- autonomous system (S), $x^+ = f(x)$, $x \in \mathcal{X}$
- autonomous linear system (SLin), $x^+ = Ax$, $x \in \mathcal{X}$,
- controlled system (CS), $x^+ = f(x,u)$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- controlled linear system (CSLin), $x^+ = Ax + Bu$, $(x,u) \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{U}$,
- Closed-loop system (CLS), $x^+ = f(x, \kappa(x))$, $(x, \kappa(x)) \in \mathcal{Z}$
- Closed-loop system (CLSLin), $x^+ = Ax + B\kappa(x)$, $(x, \kappa(x)) \in \mathcal{Z}$
- $\mathcal{X} \in \mathbb{R}^n$ closed and convex, and $\mathcal{U} \in \mathbb{R}^m$ and compact (closed and bounded) and convex.
- We will denote the solution to (S), as $\phi(k;x)$, $k \geq 0$, for an initial state $\phi(0;x) = x$, and the solution to (CS), as $\phi(k;x,\mathbf{u})$, $k \geq 0$, for an initial state $\phi(0;x,\mathbf{u}) = x$ and an input sequence $\mathbf{u} = \{u(0), \cdots, u(k-1)\}$.

Equilibrium and Invariant Sets

Equilibrium Point

A point $x_s \in \mathcal{X}$ is an equilibrium point of (S) if $x_s = f(x_s)$.

Equilibrium and Invariant Sets

Equilibrium Point

A point $x_s \in \mathcal{X}$ is an equilibrium point of (S) if $x_s = f(x_s)$.

Control Equilibrium Set

A set $\Omega \subseteq \mathcal{X}$ is a Control Equilibrium Set of (CS) if for every point $x \in \Omega$ it follows that $x = f(x, u)$, for some $u \in \mathbb{U}$.
A set $\Omega \subseteq \mathcal{X}$ is a control equilibrium set for the closed-loop system (CLS) if for every $x \in \Omega$ it follows that $x = f(x, \kappa(x))$ and $\kappa(x) \in \mathcal{U}$.

Equilibrium and Invariant Sets

Equilibrium Point

A point $x_s \in \mathcal{X}$ is an equilibrium point of (S) if $x_s = f(x_s)$.

Control Equilibrium Set

A set $\Omega \subseteq \mathcal{X}$ is a Control Equilibrium Set of (CS) if for every point $x \in \Omega$ it follows that $x = f(x, u)$, for some $u \in \mathbb{U}$.
A set $\Omega \subseteq \mathcal{X}$ is a control equilibrium set for the closed-loop system (CLS) if for every $x \in \Omega$ it follows that $x = f(x, \kappa(x))$ and $\kappa(x) \in \mathcal{U}$.

Invariant Set

A set $\Omega \subseteq \mathcal{X}$ is an invariant set for (S) if $x \in \Omega$ implies that $f(x) \in \Omega$.

Equilibrium and Invariant Sets

Equilibrium Point

A point $x_s \in \mathcal{X}$ is an equilibrium point of (S) if $x_s = f(x_s)$.

Control Equilibrium Set

A set $\Omega \subseteq \mathcal{X}$ is a Control Equilibrium Set of (CS) if for every point $x \in \Omega$ it follows that $x = f(x,u)$, for some $u \in \mathbb{U}$.
A set $\Omega \subseteq \mathcal{X}$ is a control equilibrium set for the closed-loop system (CLS) if for every $x \in \Omega$ it follows that $x = f(x, \kappa(x))$ and $\kappa(x) \in \mathcal{U}$.

Invariant Set

A set $\Omega \subseteq \mathcal{X}$ is an invariant set for (S) if $x \in \Omega$ implies that $f(x) \in \Omega$.

- The closed set $\Omega = \{x_s\}$ is a particular case of invariant set.

Equilibrium and Invariant Sets

Equilibrium Point

A point $x_s \in \mathcal{X}$ is an equilibrium point of (S) if $x_s = f(x_s)$.

Control Equilibrium Set

A set $\Omega \subseteq \mathcal{X}$ is a Control Equilibrium Set of (CS) if for every point $x \in \Omega$ it follows that $x = f(x, u)$, for some $u \in \mathbb{U}$.
A set $\Omega \subseteq \mathcal{X}$ is a control equilibrium set for the closed-loop system (CLS) if for every $x \in \Omega$ it follows that $x = f(x, \kappa(x))$ and $\kappa(x) \in \mathcal{U}$.

Invariant Set

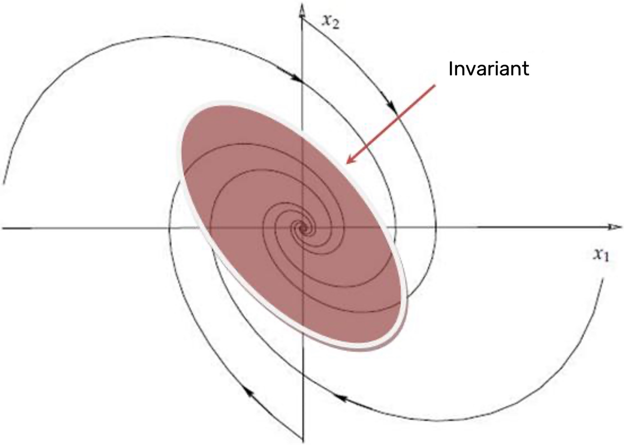
A set $\Omega \subseteq \mathcal{X}$ is an invariant set for (S) if $x \in \Omega$ implies that $f(x) \in \Omega$.

- The closed set $\Omega = \{x_s\}$ is a particular case of invariant set.

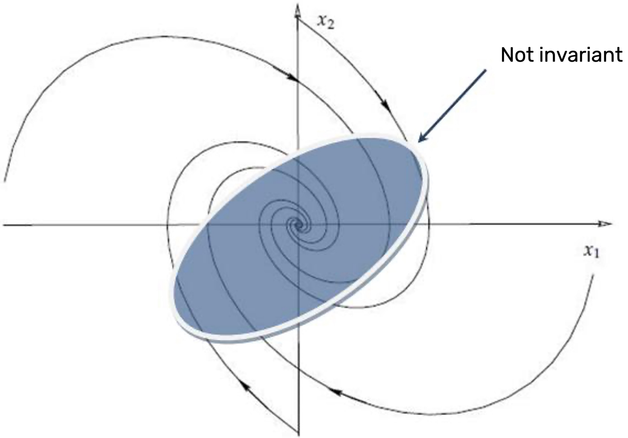
Control invariant Set

A set $\Omega \subseteq \mathcal{X}$ is a Control Invariant Set for the controlled system (CS) if $x \in \Omega$ implies that $f(x, u) \in \Omega$ for some $u \in \mathcal{U}$.

Invariant Set in $\mathbb{R}^2$



Not invariant set in $\mathbb{R}^2$



Attractivity and Stability

The following definitions are referred to systems (S) y (CS).

Attractivity and Stability

The following definitions are referred to systems (S) y (CS).

Local Stability

The (closed and invariant) set Ω is locally stable for (S) if, for all $\epsilon \geq 0$, there exists a $\delta > 0$ such that $|x|_\Omega < \delta$ implies $|\phi(i; x)|_\Omega < \epsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Attractivity and Stability

The following definitions are referred to systems (S) y (CS).

Local Stability

The (closed and invariant) set Ω is locally stable for (S) if, for all $\epsilon \geq 0$, there exists a $\delta > 0$ such that $|x|_\Omega < \delta$ implies $|\phi(i;x)|_\Omega < \epsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Global Attractivity

The (closed and invariant) set Ω is globally attractive for (S) if, $|\phi(i;x)|_\Omega \to 0$ as $i \to \infty$, for all $x \in \mathbb{R}^n$.

Attractivity and Stability

The following definitions are referred to systems (S) y (CS).

Local Stability

The (closed and invariant) set Ω is locally stable for (S) if, for all $\epsilon \geq 0$, there exists a $\delta > 0$ such that $|x|_\Omega < \delta$ implies $|\phi(i;x)|_\Omega < \epsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

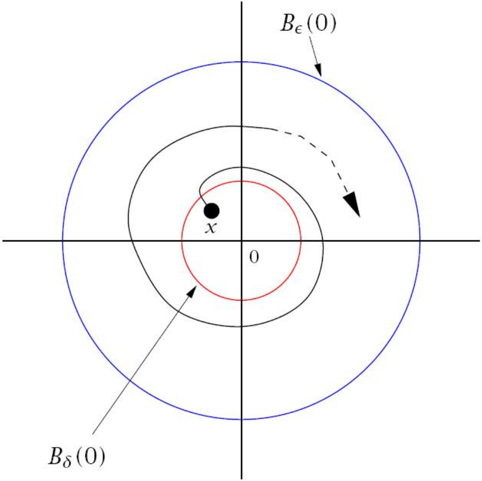
Global Attractivity

The (closed and invariant) set Ω is globally attractive for (S) if, $|\phi(i;x)|_\Omega \to 0$ as $i \to \infty$, for all $x \in \mathbb{R}^n$.

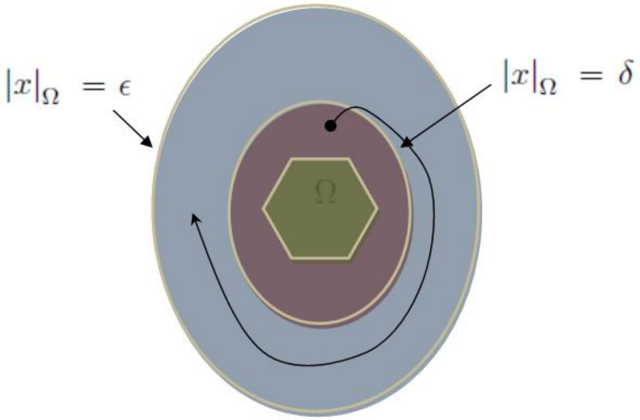
Global asymptotic stability (GAS)

The (closed and invariant) set Ω is globally asymptotically stable for (S) if it is locally stable and globally attractive.

Stability at the origin



Stability of an invariant set Ω



Attractivity and Stability

In practice GAS cannot be achieved due to the constraints $x \in \mathcal{X}$

Attractivity and Stability

In practice GAS cannot be achieved due to the constraints $x \in \mathcal{X}$

Local Stability

The (closed and invariant) set Ω is locally stable for (S) if, for all $\epsilon \geq 0$, there exists a $\delta > 0$ such that $|x|_\Omega < \delta$ implies $|\phi(i;x)|_\Omega < \epsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Attractivity and Stability

In practice GAS cannot be achieved due to the constraints $x \in \mathcal{X}$

Local Stability

The (closed and invariant) set Ω is locally stable for (S) if, for all $\epsilon \geq 0$, there exists a $\delta > 0$ such that $|x|_\Omega < \delta$ implies $|\phi(i;x)|_\Omega < \epsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Attractivity

The (closed and invariant) set Ω is attractive for (S) if, $|\phi(i;x)|_\Omega \to 0$ as $i \to \infty$, for all $x \in \mathcal{X}$.

Attractivity and Stability

In practice GAS cannot be achieved due to the constraints $x \in \mathcal{X}$

Local Stability

The (closed and invariant) set Ω is locally stable for (S) if, for all $\epsilon \geq 0$, there exists a $\delta > 0$ such that $|x|_\Omega < \delta$ implies $|\phi(i;x)|_\Omega < \epsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Attractivity

The (closed and invariant) set Ω is attractive for (S) if, $|\phi(i;x)|_\Omega \to 0$ as $i \to \infty$, for all $x \in \mathcal{X}$.

Asymptotic stability (AS)

The (closed and invariant) set Ω is asymptotically stable for (S) if it is locally stable and attractive.

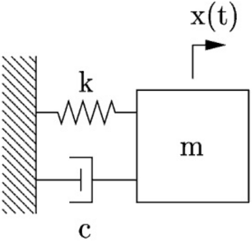
Lyapunov Stability Theory

Lyapunov theory

- Lyapunov theory is the mathematical extension of a physical observation: if a physical system dissipates mechanical energy, then it eventually settles down to an equilibrium point.

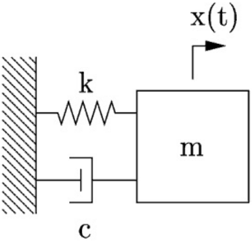
Lyapunov theory

- Lyapunov theory is the mathematical extension of a physical observation: if a physical system dissipates mechanical energy, then it eventually settles down to an equilibrium point.
- Consider the mass-spring-damper system



Lyapunov theory

- Lyapunov theory is the mathematical extension of a physical observation: if a physical system dissipates mechanical energy, then it eventually settles down to an equilibrium point.
- Consider the mass-spring-damper system



- Dynamic equation

$$m\ddot{x} + c\dot{x} + kx = 0$$

Lyapunov theory

- Total mechanical energy = kinetic energy + potential energy

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (kx)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Lyapunov theory

- Total mechanical energy = kinetic energy + potential energy

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (kx)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Zero energy corresponds to the equilibrium $(x = 0,\ \dot{x} = 0)$.
- Asymptotic stability implies that energy converges to zero.
- Instability is related to the growth of mechanical energy.

Lyapunov theory

- Total mechanical energy = kinetic energy + potential energy

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (kx)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Zero energy corresponds to the equilibrium ($x = 0$, $\dot{x} = 0$).
- Asymptotic stability implies that energy converges to zero.
- Instability is related to the growth of mechanical energy.
- Rate of energy during system's motion:

$$\begin{aligned} \dot{V}(x) &= m\ddot{x}\dot{x} + kx\dot{x} = (-c\dot{x})\dot{x} \\ &= -c\dot{x}^2 \end{aligned}$$

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$
- $c > 0$ implies that the energy of the system is continuously dissipated by the damper until the mass settles down $(\dot{x} = 0)$: AS.

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$
- $c > 0$ implies that the energy of the system is continuously dissipated by the damper until the mass settles down $(\dot{x} = 0)$: AS.
- $c = 0$ implies that the energy of the system does not dissipate or grow. We have harmonic oscillations: only stability for bounded initial conditions.

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$
- $c > 0$ implies that the energy of the system is continuously dissipated by the damper until the mass settles down $(\dot{x} = 0)$: AS.
- $c = 0$ implies that the energy of the system does not dissipate or grow. We have harmonic oscillations: only stability for bounded initial conditions.
- $c < 0$ implies that the energy of the system grows and the mass never settles down: instability.

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$
- $c > 0$ implies that the energy of the system is continuously dissipated by the damper until the mass settles down ($\dot{x} = 0$): AS.
- $c = 0$ implies that the energy of the system does not dissipate or grow. We have harmonic oscillations: only stability for bounded initial conditions.
- $c < 0$ implies that the energy of the system grows and the mass never settles down: instability.
- Finding an energy function with negative rate is a sufficient condition for AS.

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$
- $c > 0$ implies that the energy of the system is continuously dissipated by the damper until the mass settles down ($\dot{x} = 0$): AS.
- $c = 0$ implies that the energy of the system does not dissipate or grow. We have harmonic oscillations: only stability for bounded initial conditions.
- $c < 0$ implies that the energy of the system grows and the mass never settles down: instability.
- Finding an energy function with negative rate is a sufficient condition for AS.
- This function is what we call a Lyapunov function.

Lyapunov function

- Rate of energy during system's motion: $\dot{V}(x) = -c\dot{x}^2$
- $c > 0$ implies that the energy of the system is continuously dissipated by the damper until the mass settles down $(\dot{x} = 0)$: AS.
- $c = 0$ implies that the energy of the system does not dissipate or grow. We have harmonic oscillations: only stability for bounded initial conditions.
- $c < 0$ implies that the energy of the system grows and the mass never settles down: instability.
- Finding an energy function with negative rate is a sufficient condition for AS.
- This function is what we call a Lyapunov function.

In what follows we will make use of some support function in order to provide a mathematical definition of Lyapunov function.

Support functions

Consider the following definitions:

Support functions

Consider the following definitions:

Function $\mathcal{K}$

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if:

- it is continuous.
- it is strictly increasing, i.e., if $a > b$, then $\alpha(a) > \alpha(b)$.
- $\alpha(0) = 0$.

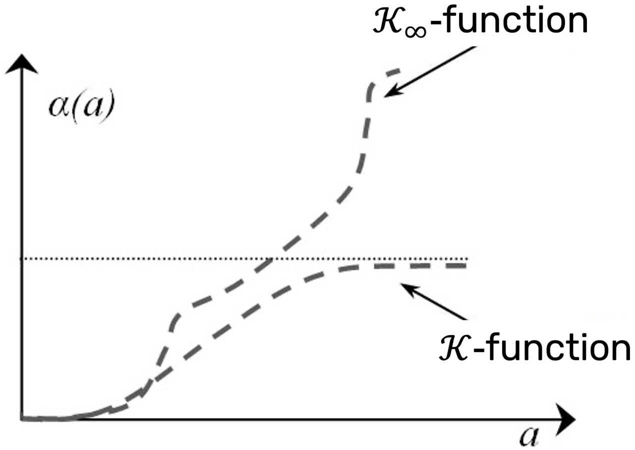
A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and

- $\alpha(a) \to \infty$ when $a \to \infty$ (unbounded).

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function of:

- the function $\beta(a, k)$ is $\mathcal{K}$ in a for every fixed $k \geq 0$.
- the function $\beta(a, k)$ is nonincreasing in k for every fixed $a \geq 0$, in such a way that $\beta(a, k) \to 0$ for $k \to \infty$.

$\mathcal{K}$ and $\mathcal{K}_\infty$-functions



Example of $\mathcal{K}$ -function

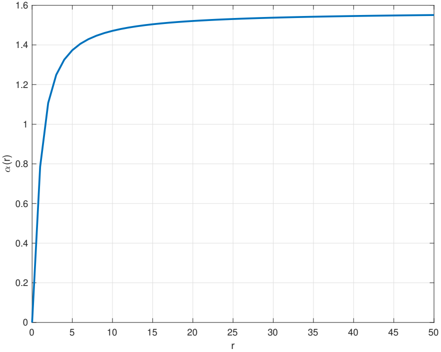
Consider the function

$$\alpha(r) = tan^{-1}(r)$$

- It is strictly increasing as

$$\frac{d\alpha(r)}{dr} = \frac{1}{(1+r)^2} > 0$$

- $\alpha(0) = 0$
- $\lim_{r\to\infty} \alpha(r) = \frac{\pi}{2}$



It's a $\mathcal{K}$ -function.

Example of $\mathcal{K}_\infty$ -function

Consider the function

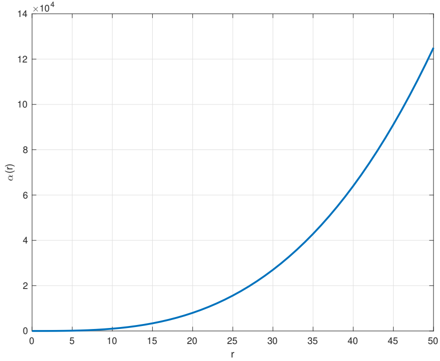
$$\alpha(r) = r^c$$

for any $c > 0$

- It is strictly increasing as

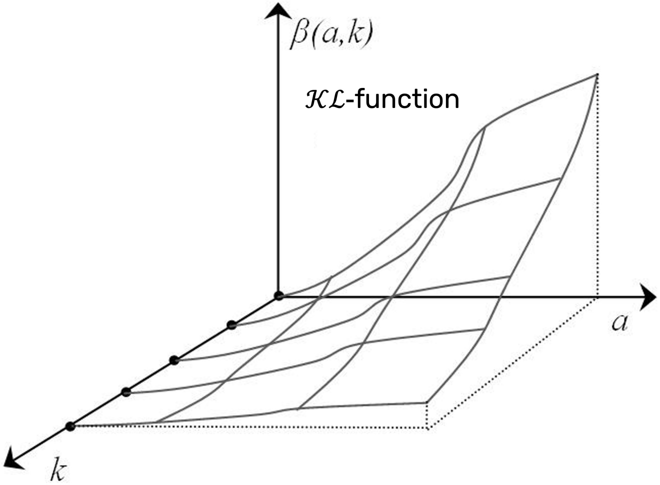
$$\frac{d\alpha(r)}{dr} = cr^{c-1} > 0$$

- $\alpha(0) = 0$
- $\lim_{r\to\infty} \alpha(r) = \infty$



It's a $\mathcal{K}_\infty$ -function.

$\mathcal{KL}$-function



Example of $\mathcal{KL}$ -function

Consider the function

$$\beta(r,s) = \frac{r}{(krs+1)}, \quad k > 0$$

- It is strictly increasing in r:

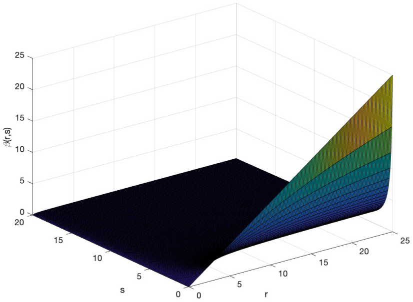
$$\frac{\partial \beta(r,s)}{\partial r} = \frac{1}{(krs+1)^2} > 0$$

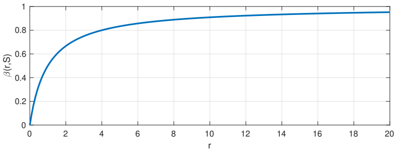
- It is strictly decreasing in s:

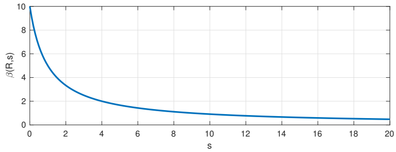
$$\frac{\partial \beta(r,s)}{\partial s} = \frac{-kr^2}{(krs+1)^2} < 0$$

- $\beta(0,0) = 0$
- $\beta(r,s) \to 0$ as $s \to \infty$

It's a $\mathcal{KL}$ -function.







Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions), then

- $\alpha_i^{-1}(\cdot)$ defines the inverse function of $\alpha_i(\cdot)$ and it is defined in $[0, \mathbb{R})$.

Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions), then

- $\alpha_i^{-1}(\cdot)$ defines the inverse function of $\alpha_i(\cdot)$ and it is defined in $[0, \mathbb{R})$.
- $\alpha_1^{-1}(\cdot)$ and $\alpha_1 \circ \alpha_2(\cdot) = \alpha_1(\alpha_2(\cdot))$ are $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions)

Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions), then

- $\alpha_i^{-1}(\cdot)$ defines the inverse function of $\alpha_i(\cdot)$ and it is defined in $[0, \mathbb{R})$.
- $\alpha_1^{-1}(\cdot)$ and $\alpha_1 \circ \alpha_2(\cdot) = \alpha_1(\alpha_2(\cdot))$ are $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions)
- Even more, if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$-functions and $\beta(\cdot, \cdot)$ is a $\mathcal{KL}$-function, then $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is a $\mathcal{KL}$-function

Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Properties of $\mathcal{K}$ and $\mathcal{K}_\infty$-functions

Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions), then

- $\alpha_i^{-1}(\cdot)$ defines the inverse function of $\alpha_i(\cdot)$ and it is defined in $[0, \mathbb{R})$.
- $\alpha_1^{-1}(\cdot)$ and $\alpha_1 \circ \alpha_2(\cdot) = \alpha_1(\alpha_2(\cdot))$ are $\mathcal{K}$-functions ($\mathcal{K}_\infty$-functions)
- Even more, if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$-functions and $\beta(\cdot, \cdot)$ is a $\mathcal{KL}$-function, then $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is a $\mathcal{KL}$-function

Positive definite function

A function $V : \mathbb{R}^{nx} \to \mathbb{R}_{\geq 0}$ is locally positive definite ($\mathcal{PD}$) if it is continuous, $V(0) = 0$ and $V(x) > 0$ for every $x \neq 0$ in a neighborhood or the origin.

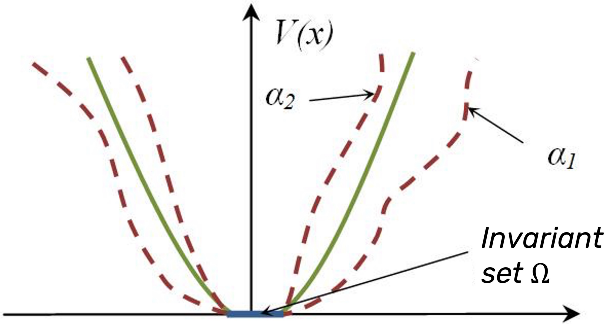
Lyapunov Stability Theory

Definition (Lyapunov function)

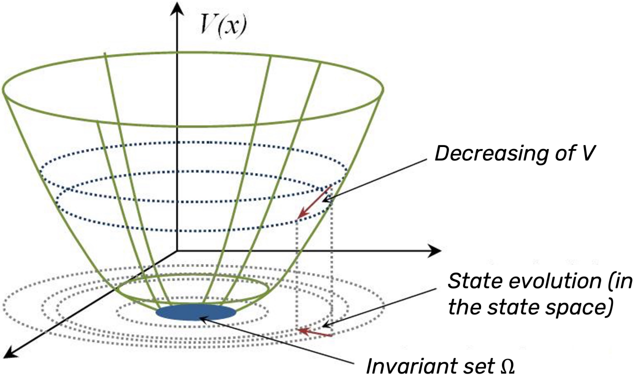
A function $V : \mathbb{R}^{nx} \to \mathbb{R}_{\geq 0}$ *is said to be a Lyapunov function for the system* $x^+ = f(x)$ *and set* Ω *if there exist* $\mathcal{K}_\infty$*-functions* α_1 *y* α_2*, and a* $\mathcal{PD}$ *function* α_3*, such that for any* $x \in \mathbb{R}^{nx}$*,*

$$\begin{aligned} V(x) &\geq \alpha_1(|x|_\Omega), \\ V(x) &\leq \alpha_2(|x|_\Omega), \\ V(f(x)) - V(x) &\leq -\alpha_3(|x|_\Omega), \end{aligned}$$

Lyapunov function in $\mathbb{R}$



Lyapunov function in $\mathbb{R}^2$



Lyapunov Stability Theory

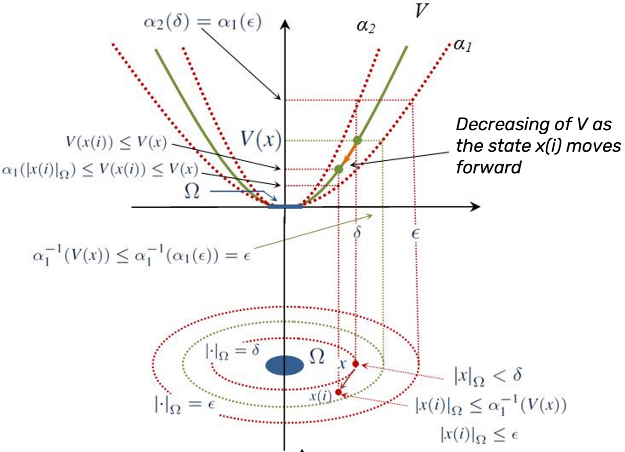
The existence of a Lyapunov function is a sufficient condition for (global) asymptotic stability as shown in the next result which can be proved under the assumption, common in MPC, that $\alpha_3(\cdot)$ is $\mathcal{K}_\infty$.

Theorem (Lyapunov function and GAS)

Suppose $V(\cdot)$ is a Lyapunov function for $x^+ = f(x)$ and set Ω, with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$-function. Then Ω is globally asymptotically stable.

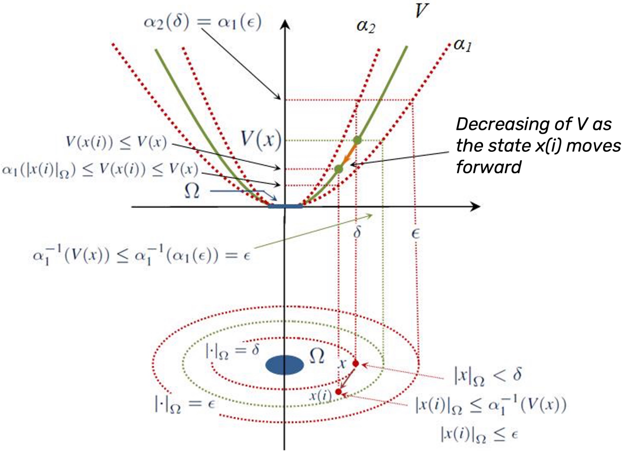
A detailed proof can be found in (Rawlings & Mayne, 2009, Appendix B, Theorem B.11, pag. 609)

Proof (Stability):



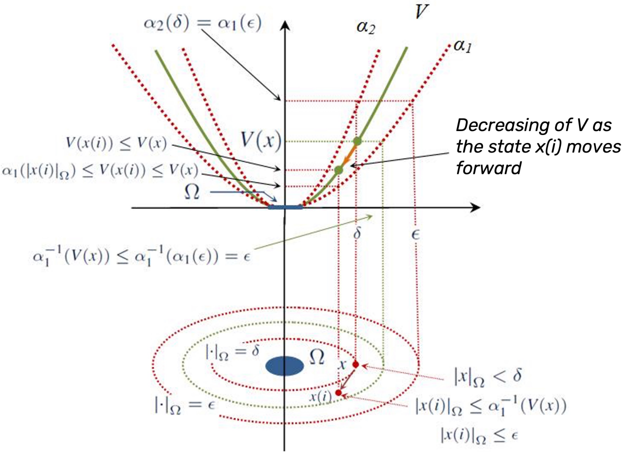
- Let $\epsilon > 0$ be arbitrary and let $\delta \triangleq \alpha_2^{-1}(\alpha_1(\epsilon))$.

Proof (Stability):



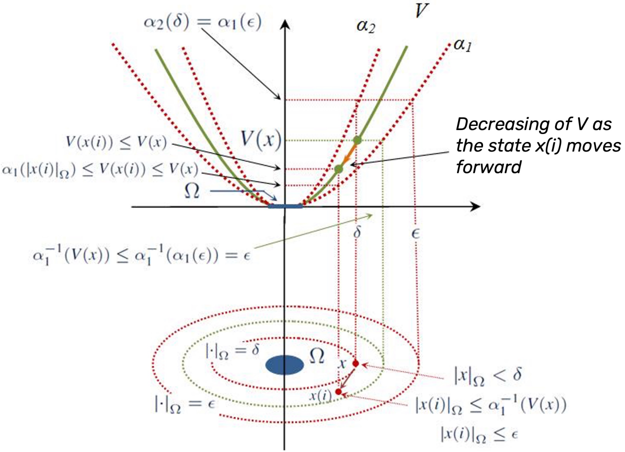
- Suppose x is the initial state, and $|x|_\Omega < \delta$.

Proof (Stability):



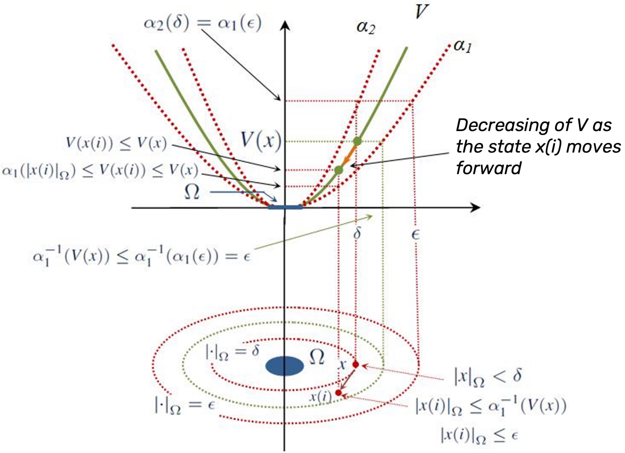
- From the second condition, $V(x) \leq \alpha_2(|x|_\Omega)$, then $V(x) \leq \alpha_2(\delta) = \alpha_1(\epsilon)$.

Proof (Stability):



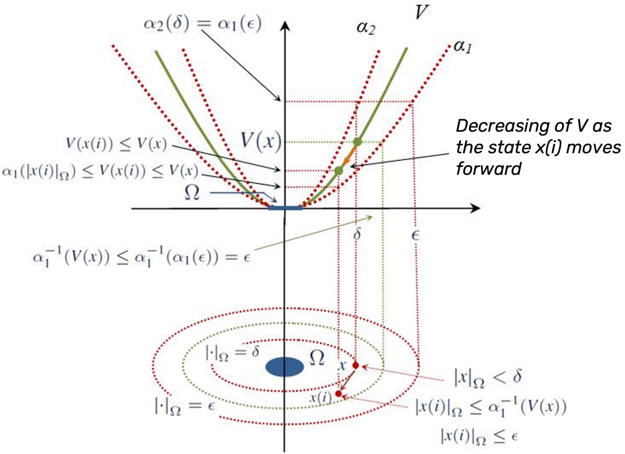
- From $V(f(x)) - V(x) \leq -\alpha_3(|x|_\Omega)$, then $\{V(x(i)) : i \in \mathbb{I}_{\geq 0}\}$, with $x(i) \triangleq \phi(i;x)$, is a nonincreasing sequence.

Proof (Stability):



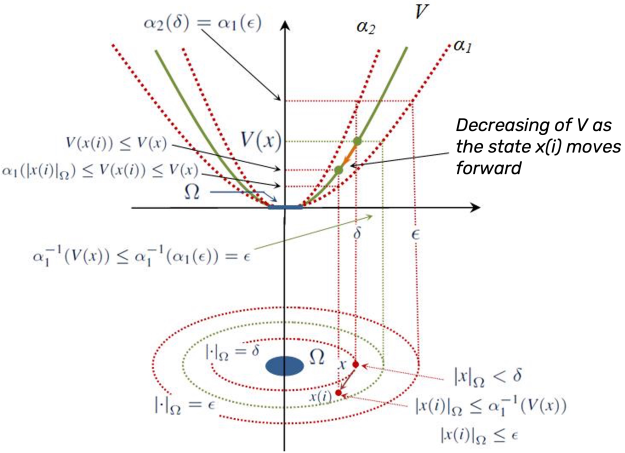
- So, for all $i \in \mathbb{I}_{\geq 0}$ $V(x(i)) \leq V(x)$, where x is the initial state.

Proof (Stability):



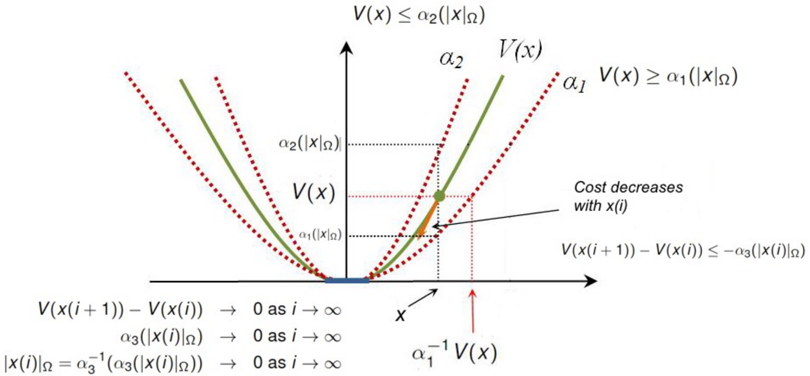
- From $V(x) \geq \alpha_1(|x|_\Omega)$, then $\alpha_1(|x(i)|_\Omega) \leq V(x(i)) \leq V(x)$. Recall that $V(x) \leq \alpha_2(\delta) = \alpha_1(\epsilon)$.

Proof (Stability):



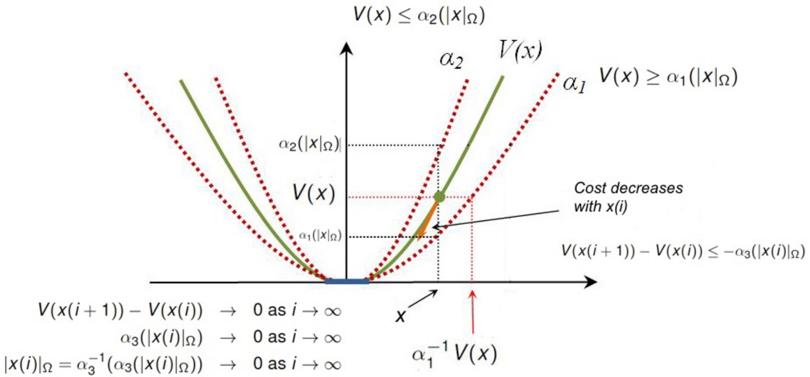
- Then $|x(i)|_\Omega \leq \alpha_1^{-1}(V(x)) \leq \alpha_1^{-1}(\alpha_1(\epsilon)) = \epsilon$, for any $|x|_\Omega \leq \delta$

Proof (Attractivity/Convergence):



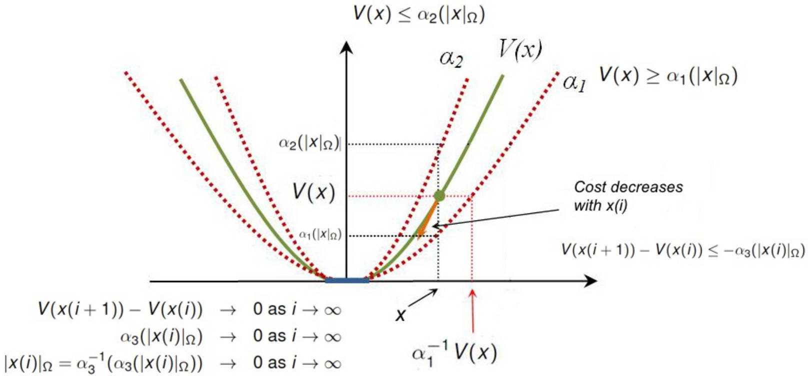
- Let $x \in \mathbb{R}^{nx}$ be arbitrary. From the second Lyapunov function condition $V(x)$ is finite, and from the first and third condition, $\{V(x(i)) : i \in \mathbb{I}_{\geq 0}\}$, with $x(i) \triangleq \phi(i; x)$, is nonincreasing and bounded from below by zero.

Proof (Attractivity/Convergence):



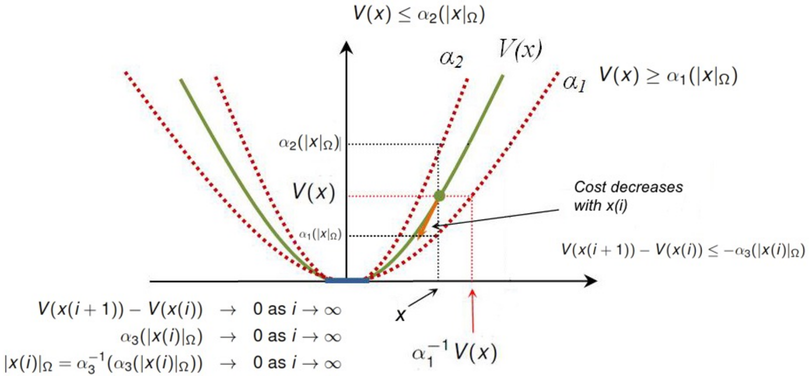
- Hence both, $V(x(i))$ and $V(x(i+1))$ converge to a $\bar{V} \geq 0$, as $i \rightarrow \infty$ (prop. of real sequences).

Proof (Attractivity/Convergence):



- Since $[V(x(i+1)) - V(x(i))] \rightarrow 0$ as $i \rightarrow \infty$ and $x(i+1) = f(x(i))$ then from the third condition $\alpha_3(|x(i)|_\Omega) \rightarrow 0$ as $i \rightarrow \infty$.

Proof (Attractivity/Convergence):



- Since $|x(i)|_\Omega = \alpha_3^{-1}(\alpha_3(|x(i)|_\Omega))$, where α_3^{-1} is a $\mathcal{K}_\infty$-function, then $|x(i)|_\Omega \to 0$ as $i \to \infty$.

Stable MPC

- We use Lyapunov stability theory.

Stable MPC

- We use Lyapunov stability theory.
- The objective is then to find a Lyapunov function $V(\cdot)$ for the closed-loop system under MPC, $x^+ = f(x, \kappa_N(x))$.

Stable MPC

- We use Lyapunov stability theory.
- The objective is then to find a Lyapunov function $V(\cdot)$ for the closed-loop system under MPC, $x^+ = f(x, \kappa_N(x))$.
- Standard method to ensure stability: **use of the optimal cost function as a candidate Lyapunov function**.

Stable MPC

- We use Lyapunov stability theory.
- The objective is then to find a Lyapunov function $V(\cdot)$ for the closed-loop system under MPC, $x^+ = f(x, \kappa_N(x))$.
- Standard method to ensure stability: **use of the optimal cost function as a candidate Lyapunov function**.
- We need to chose appropriately the ingredients of the controller: stage cost, terminal constraint, terminal cost.

$$
\begin{aligned}
V_N^0(x) = \quad & \min_{\mathbf{u}} \sum_{j=0}^{N-1} \ell(x(j), u(j)) \\
& s.t. \quad x(0) = x, \quad x(j+1) = f(x(j), u(j)) \\
& \qquad x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \\
& \qquad x(N) = 0
\end{aligned}
$$

Stable MPC

- We use Lyapunov stability theory.
- The objective is then to find a Lyapunov function $V(\cdot)$ for the closed-loop system under MPC, $x^+ = f(x, \kappa_N(x))$.
- Standard method to ensure stability: **use of the optimal cost function as a candidate Lyapunov function**.
- We need to chose appropriately the ingredients of the controller: stage cost, terminal constraint, terminal cost.

$$
\begin{aligned}
V_N^0(x) = \quad \min_{\mathbf{u}} \quad & \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N)) \\
s.t. \quad & x(0) = x, \quad x(j+1) = f(x(j), u(j)) \\
& x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \\
& x(N) \in \mathbb{X}_f
\end{aligned}
$$

Main References

Most the presented concepts were taken from: Goodgwin et al. (2005), Rawlings & Mayne (2009), Blanchini & Miani (2008) and Khalil (1996).

Blanchini, F., & Miani, S. (2008). *Set-Theoretic Method in Control*. Birkhäuser.

Boyd, S., & Vandenberghe, L. (2006). *Convex Optimization*. Cambridge University Press.

Camacho, E. F., & Bordons, C. (2004). *Model Predictive Control*. Springer-Verlag, 2nd ed.

Goodgwin, G., Seron, M., & de Dona, J. (2005). *Constrained control and estimation. An optimisation approach*. Springer.

Khalil, H. (1996). *Nonlinear Systems*. Prentice-Hall, 2 ed.

Limon, D., Alamo, T., Raimondo, D. M., de la Peña, D. M., Bravo, J. M., Ferramosca, A., & Camacho, E. F. (2009). Input-to-state stability: an unifying framework for robust model predictive control. In L. Magni, D. M. Raimondo, & F. Allgöwer (Eds.) *International Workshop on Assessment and Future Direction of Nonlinear Model Predictive Control*, (pp. 1–26). Springer.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, *36*, 789–814.

Rawlings, J. B., & Mayne, D. Q. (2009). *Model Predictive Control: Theory and Design*. Nob-Hill Publishing, 1st ed.