Fundamentals of Model Predictive Control

Economic MPC

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Università degli Studi di Bergamo

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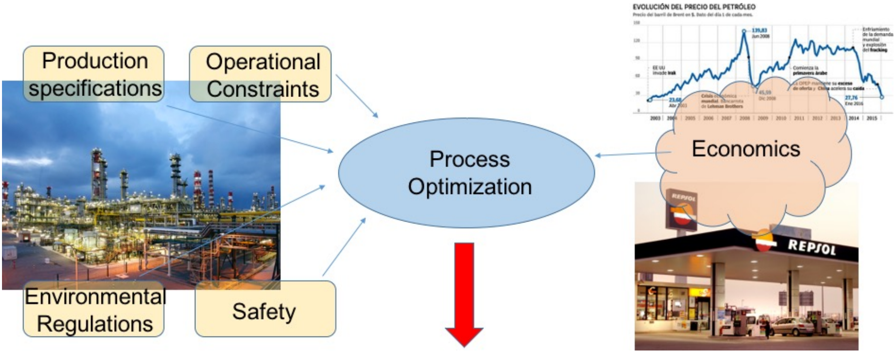
Outline

A motivational example

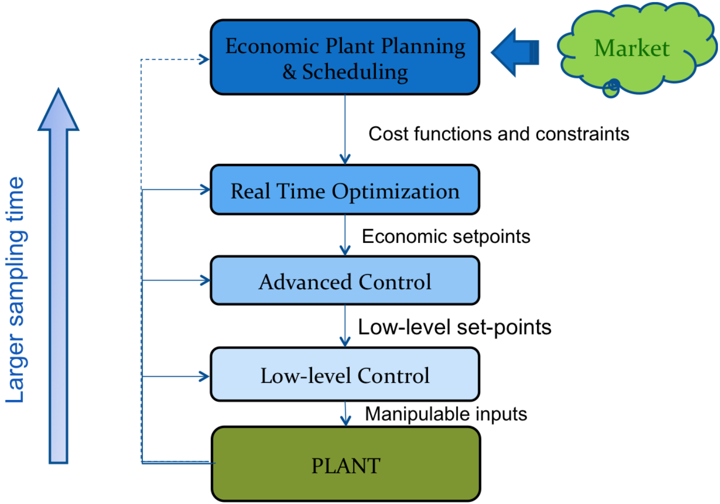
Gas and Oil value chain

A motivational example



Higher Profit

Hierarchical control structure



RTO problem

Find the economically optimal setpoint:

Definition

The economically optimal setpoint, (x_{sp}, u_{sp}), satisfy

$$
\begin{aligned}
(x_{sp}, u_{sp}) \quad = \quad & \arg\min_{x,u} \ell(x,u) \\
s.t. \quad & F_p(x,u) = 0 \\
& x \in \mathcal{X}, \quad u \in \mathcal{U}
\end{aligned}
$$

- $\ell(x,u)$ plant profit function.
- $F_p(x,u)$ complex continuous nonlinear model of the plant.

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RTO vs MPC

RTO:

- Model-based optimizer.
- Operated in closed-loop.
- Static optimization.
- Complex continuous nonlinear model of the plant: $\dot{x} = F_p(x, u)$.
- Time scale: hours or days.

MPC:

- Model-based optimizer.
- Operated in closed-loop.
- Dynamic optimization.
- Simple discrete model of the plant: $x^+ = f(x, u)$ or $x^+ = Ax + Bu$.
- Sampling time: minutes.

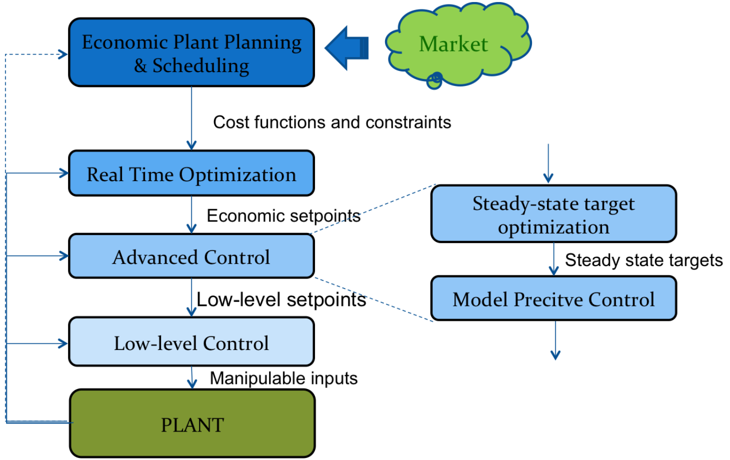
RTO vs MPC

(x_{sp}, u_{sp}) may be inconsistent/unreachable for the MPC layer.

- Different time scale.
- Different models.

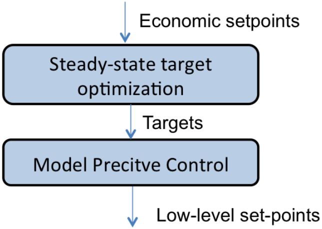
Solution: adding a new optimization level in between of RTO and MPC, referred as the Steady State Target Optimizer (SSTO) (Muske, 1997; Rao & Rawlings, 1999; Ying & Joseph, 1999).

Hierarchical control structure: SSTO



Steady State Target Optimizer

Steady State Optimizer: For a given RTO setpoint (x_{sp}, u_{sp}), computes an admissible steady state target for the MPC (x_s^*, u_s^*), typically minimizing a linear or quadratic error function (also knonw as LP-MPC or QP-MPC structure).



Quadratic SSTO

$$\begin{aligned}(x_s^*, u_s^*) &= \arg\min_{x,u} \|x - x_{sp}\|_M^2 + \|u - u_{sp}\|_T^2 \\ s.t. \quad & x = Ax + Bu \\ & x \in \mathcal{X}, \quad u \in \mathcal{U}\end{aligned}$$

- M and T positive definite.

(x_s^*, u_s^*) is now admissible/reachable for the MPC.

Tracking MPC: terminal equality constraint

Definition (MPC cost function)

$$V_N(x;\mathbf{u}) = \sum_{j=0}^{N-1}(\|x(j) - x_s^*\|_Q^2 + \|u(j) - u_s^*\|_R^2)$$

$Q \geq 0$ *and* $R > 0$.

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Definition (Optimization problem)

$$\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x;\mathbf{u}) \\
s.t. \quad & x(0) = x,\ x(j+1) = Ax(j) + Bu(j) \\
& x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \\
& x(N) = x_s^*
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- Optimal solution: $V_N^0(x)$ and $\mathbf{u}^0(x) = \{u^0(0;x), \dots, u^0(N-1;x)\}$.
- Receding horizon: only $\kappa_N(x) = u^0(0;x)$ is applied; then the prediction window is moved one step ahead.

Controllability and Admissible set

Assumption

The prediction horizon N is such that

$$rank(Co_N) \geq n$$

where $Co_N = [A^{N-1}B \ldots AB\,B]$ *is the N-controllability matrix of system* (A, B).

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Definition

The admissible set (or feasible region or domain of attraction) of the Tracking-MPC controller is given by

$$\mathcal{X}_N(x_s^*) = \{x \in \mathcal{X} \mid \exists \mathbf{u} \in \mathcal{U}^N \quad s.t. \quad x(j) \in \mathcal{X}, u(j) \in \mathcal{U}, j = 0, ..., N-1 \\ and\; x(N) = x_s^*\}$$

Tracking MPC: nonlinear model

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$Q \geq 0$ *and* $R > 0$.

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$$\begin{aligned} \min_{\mathbf{u}} \quad & V_N(x;\mathbf{u}) \\ s.t. \quad & x(0) = x,\ x(j+1) = f(x(j), u(j)) \\ & x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \\ & x(N) = x_s^* \end{aligned}$$

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Controllability and Admissible set

Assumption

1. *$f(.,.)$ is continuous.*
2. *(Weak controllability): there exists a $\mathcal{K}_\infty$ function γ s. t. for all $x \in \mathcal{X}_N$ there exists a feasible $\mathbf{u}$ s.t.*

$$\sum_{j=0}^{N-1} \|u(j) - u_s^*\| \leq \gamma(\|x - x_s^*\|)$$

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Example: CSTR

Isothermal CSTR with consecutive-competitive reactions (Rawlings et al., 2012).

$$\begin{aligned} P_0 + B &\rightarrow P_1 \\ P_1 + B &\rightarrow P_2 \end{aligned}$$

From dimensionless mass balance:

$$\begin{aligned} \dot{x}_1 &= u_1 - x_1 - \sigma_1 x_1 x_2 \\ \dot{x}_2 &= u_2 - x_2 - \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_3 &= -x_3 + \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_4 &= -x_4 + \sigma_2 x_2 x_3 \end{aligned}$$

x_1, x_2, x_3, x_4: concentrations of P_0, B, P_1, P_2. $0 \leq x_i \leq 10$
u_1, u_2: inflow rates of P_0 and B. $0 \leq u_i \leq 10$
$\sigma_1 = 1, \sigma_2 = 0.4, T_s = 0.1$

Example: CSTR

$$Q = 0.1 I_4, \qquad R = 0.1 I_2 \qquad N = 10$$

$$x_s^* = (4.9759, 1.0097, 3.5787, 1.4454), \quad u_s^* = (10, 7.4791)$$

Definition

The best admissible steady state and input, (x_s^*, u_s^*)*, satisfy*

$$\begin{aligned} (x_s^*, u_s^*) \quad &= \quad \arg\min_{x,u} \ell(x,u) \\ s.t. \quad & x = f(x,u) \\ & x \in \mathcal{X}, \quad u \in \mathcal{U} \end{aligned}$$

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$$\ell(x,u) = -\rho_1 x_3 + \rho_2(u_1 + u_2),\ \rho = (7,1)$$

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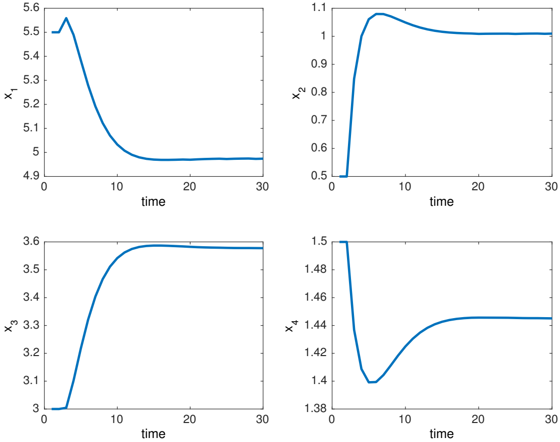

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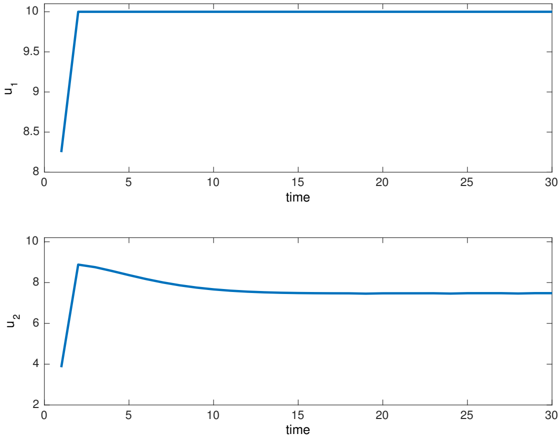

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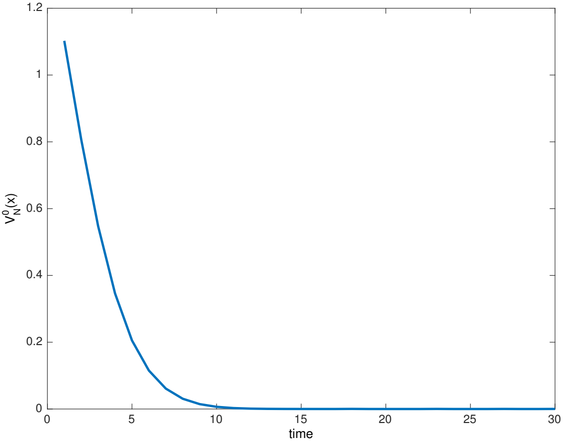

Drawback of steady state operation

Drawback of tracking MPC

- Designed to ensure asymptotic tracking of (x_s^*, u_s^*).
- Practically optimal when the operation point does not change.
- Transient cost not taken into account.
- Closed-loop behavior depends on the choice of Q and R.

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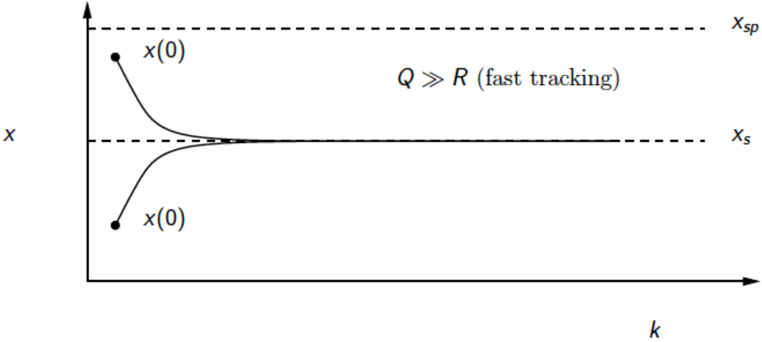

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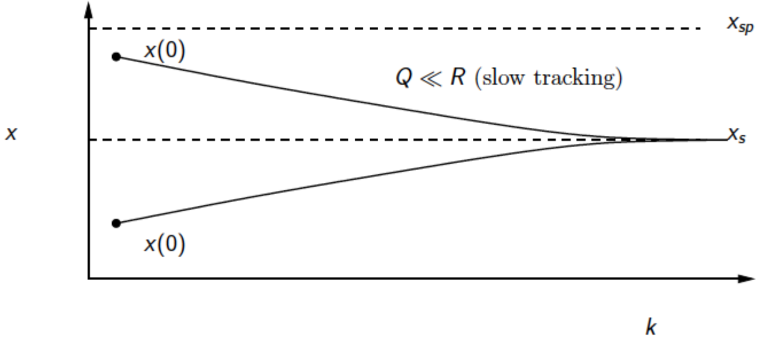

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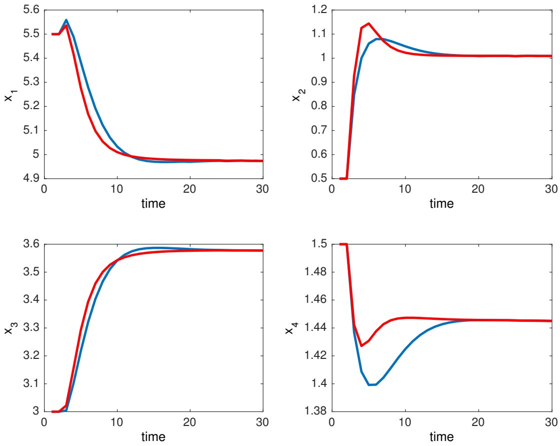

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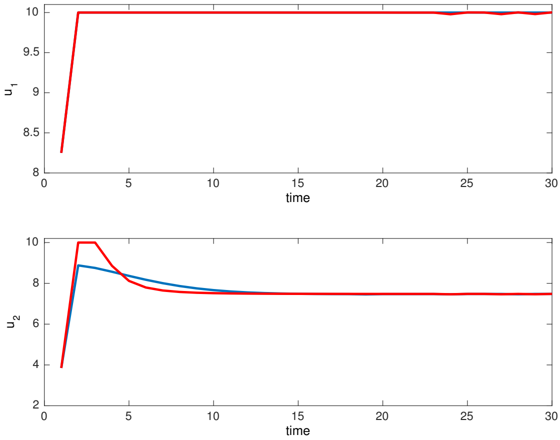

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How can we assess which setup is the best?

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We can use the economic cost:

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Why not considering it in the MPC?

Economic MPC

Is steady state operation economically optimal?

Motivation (Rawlings et al. (2012))

- *The closer the system gets to the economic optimum, the more profitable it is.*
- *Who gets closest to the global economic optimum?*
- *Tracking controllers: Rush to the target (away from non steady economic optimum).*
  - *Tracking speed chosen through penalties, but still the objective remains to drive away from non steady economic optimum!*
- *Economic optimizing controllers: Expected to get closer to the economic optimum with eventual setting at the steady target.*

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Motivation (Ellis et al. (2014))

- *Steady state operation typically adopted in chemical process industries.*

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- *Several examples of chemical reactors with periodic operation.*
- *Batch processes required to follow economically optimal trajectories.*

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- *Several examples of chemical reactors with periodic operation.*
- *Batch processes required to follow economically optimal trajectories.*

Motivation (Angeli et al. (2012))

- *Control law often designed disregarding transient costs.*
- *Plant's nonlinearities may cause the best operative regime not to be an equilibrium.*

How to achieve economic operation?

Use economic information dynamically!

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Solutions

1. *Dynamic RTO: move dynamic information in the RTO layer.*

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2. *One-layer MPC: move economic information into the control layer.*

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3. *sp-MPC: intentionally give the optimizer an unreachable setpoint.*

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1. *Dynamic RTO: move dynamic information in the RTO layer.*
2. *One-layer MPC: move economic information into the control layer.*
3. *sp-MPC: intentionally give the optimizer an unreachable setpoint.*
4. *Economic MPC: replace the MPC stage cost with the economic cost.*

How to use economic information dynamically?

1. Dynamic RTO (Biegler, 2009; Kadam & Marquardt, 2007; Würth et al., 2009)

Definition

Find the economically optimal trajectory, $(\mathbf{x}_{sp}, \mathbf{u}_{sp})$*, satisfy*

$$\begin{aligned} (\mathbf{x}_{sp}, \mathbf{u}_{sp}) &= \arg\min_{\mathbf{x},\mathbf{u}} \ell(x, u, t_f) \\ s.t. &\quad \dot{x} = F_p(x, u) \\ &\quad x \in \mathcal{X}, \quad u \in \mathcal{U} \end{aligned}$$

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- Dynamic information into RTO layer.
- It can include complex nonlinear constraint.
- Dynamic economic problem solved in slow time scale.
- It delivers economic trajectories (not setpoints).
- Batch processes, cyclic processes, periodic processes.

How to use economic information dynamically?

2. One-layer MPC (Adetola & Guay, 2010; Zanin et al., 2002)

Definition

$$\min_{\mathbf{u}} \quad \sum_{j=0}^{N-1} (\|x(j)\|_Q^2 + \|u(j)\|_R^2) + \ell(x(N), u(N))$$

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Definition

$$\min_{\mathbf{u}} \sum_{j=0}^{N-1} (\|x(j)\|_Q^2 + \|u(j)\|_R^2) + \ell(x(N), u(N))$$

- Economic information in the control layer.
- Tracking cost modified by adding an economic terminal cost.
- Controller has no knowledge of the economics prior the end of the horizon.
- Approximation by means of the gradient (De Souza et al., 2010).
- Stability proved by means of relaxed terminal constraint (Alamo et al., 2012)

How to use economic information dynamically?

3. MPC with unreachable setpoints (Rawlings et al., 2008)

Definition (sp-MPC)

$$
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s.t. \quad & x(j+1) = Ax(j) + Bu(j) \\
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- MPC cost as a tracking cost.
- Measure of the distance to (x_{sp}, u_{sp}) instead of (x_s^*, u_s^*).
- Better transient performance than tracking MPC.
- Complex stability proof based on convexity.

sp-MPC vs track-MPC

sp-MPC vs track-MPC

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix}, \; \|x\|_\infty \leq 5, \; \|u\|_\infty \leq 0.5$$

$$x_{sp} = (6, 2.5), \quad u_{sp} = (2.5, -5), \quad Q = I_2, \quad R = I_2$$
$$x_s^* = (5, 0.25), \quad u_s^* = (0.25, -0.5), \quad N = 5$$

sp-MPC vs track-MPC

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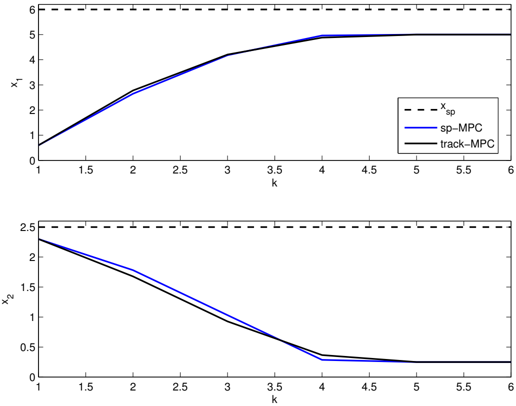

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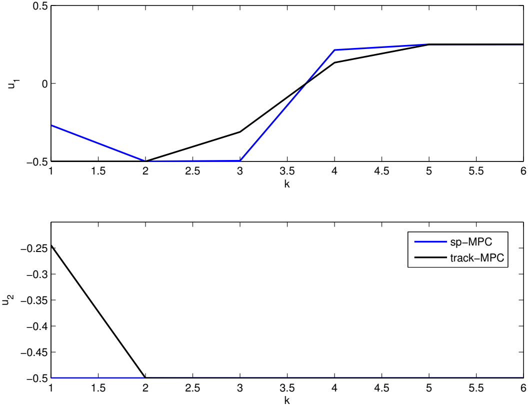

sp-MPC vs track-MPC

$$
\begin{aligned}
\Phi_x &= \frac{1}{T}\sum_{k=0}^{T}\|x(k)-x_{sp}\|_Q^2 \\
\Phi_u &= \frac{1}{T}\sum_{k=0}^{T}\|u(k)-u_{sp}\|_R^2 \\
\Phi &= \Phi_x + \Phi_u
\end{aligned}
$$

sp-MPC vs track-MPC

$$\Phi_x = \frac{1}{T}\sum_{k=0}^{T}\|x(k) - x_{sp}\|_Q^2$$

$$\Phi_u = \frac{1}{T}\sum_{k=0}^{T}\|u(k) - u_{sp}\|_R^2$$

$$\Phi = \Phi_x + \Phi_u$$

	Φ_x	Φ_u	Φ
sp-MPC	10.65	27.08	37.73
track-MPC	10.75	27.58	38.33

How to use economic information dynamically?

4. Economic MPC(Amrit, 2011; Rawlings et al., 2012)

Definition (EMPC)

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{j=0}^{N-1} \ell(x(j), u(j)) \\
s.t. \quad & x(j+1) = f(x(j), u(j)) \\
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How to use economic information dynamically?

4. Economic MPC(Amrit, 2011; Rawlings et al., 2012)

Definition (EMPC)

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- Economic RTO cost function as MPC stage cost function.
- The controller optimizes the economic cost of the process without reference to any steady state.

How to use economic information dynamically?

4. Economic MPC(Amrit, 2011; Rawlings et al., 2012)

Definition (EMPC)

$$\min_{\mathbf{u}} \quad \sum_{j=0}^{N-1} \ell(x(j), u(j))$$

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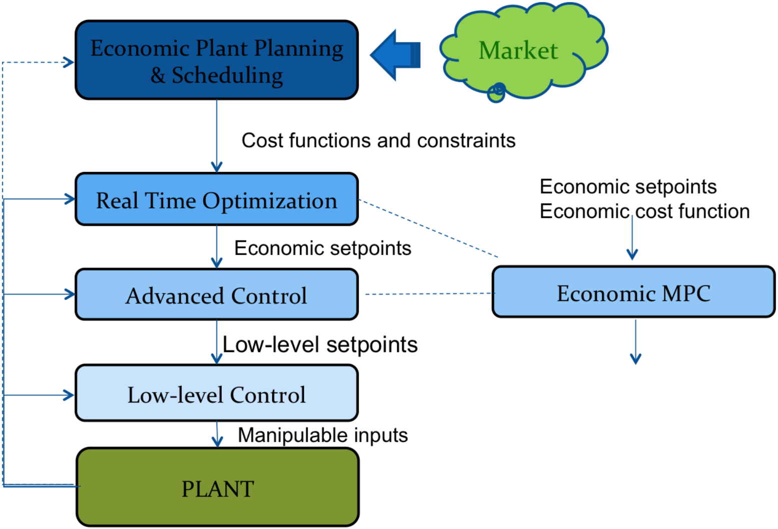
- Economic RTO cost function as MPC stage cost function.
- The controller optimizes the economic cost of the process without reference to any steady state.
- Idea first considered in the field of economics in the 1920s

"A mathematical theory of saving", The Economic Journal, (Ramsey, 1928).

Economic MPC control structure



Outline

Problem statement

Definition (System and constraints)

Consider the nonlinear discrete time system

$$x^+ = f(x,u)$$

subject to $x \in \mathcal{X} \subset \mathbb{R}^n$ *and* $u \in \mathcal{U} \subset \mathbb{R}^m$. $f(x,u)$ *continuous.*

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Definition

The best admissible steady state and input, (x_s^*, u_s^*), *satisfy*

$$\begin{aligned} (x_s^*, u_s^*) \quad &= \quad \arg\min_{x,u} \ell(x,u) \\ s.t. \quad & x = f(x,u) \\ & x \in \mathcal{X}, \quad u \in \mathcal{U} \end{aligned}$$

Economic MPC: formulation

Definition (EMPC cost function)

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- Optimal solution: $V_N^0(x)$ and $\mathbf{u}^0(x) = \{u^0(0;x), ..., u^0(N-1;x)\}$.
- Receding horizon: only $\kappa_N(x) = u^0(0;x)$ is applied; then the prediction window is moved one sted ahead.

Example: CSTR

Isothermal CSTR with consecutive-competitive reactions (Rawlings et al., 2012).

$$\begin{aligned} P_0 + B &\rightarrow P_1 \\ P_1 + B &\rightarrow P_2 \end{aligned}$$

From dimensionless mass balance:

$$\begin{aligned} \dot{x}_1 &= u_1 - x_1 - \sigma_1 x_1 x_2 \\ \dot{x}_2 &= u_2 - x_2 - \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_3 &= -x_3 + \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_4 &= -x_4 + \sigma_2 x_2 x_3 \end{aligned}$$

x_1, x_2, x_3, x_4: concentrations of P_0, B, P_1, P_2. $0 \leq x_i \leq 10$
u_1, u_2: inflow rates of P_0 and B. $0 \leq u_i \leq 10$
$\sigma_1 = 1$, $\sigma_2 = 0.4$, $T_s = 0.1$

Example: CSTR

$$\ell(x,u) = -\rho_1 x_3 + \rho_2(u_1+u_2), \qquad \rho = (7,1), \qquad N = 10$$

$$x_s^* = (4.9759, 1.0097, 3.5787, 1.4454), \quad u_s^* = (10, 7.4791)$$

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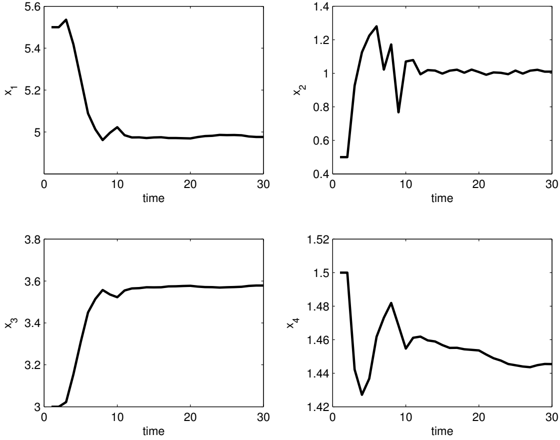

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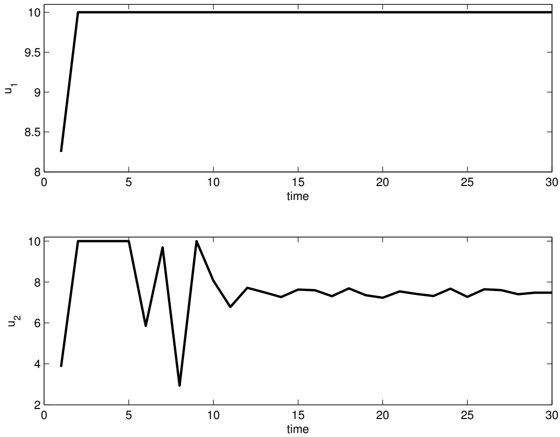

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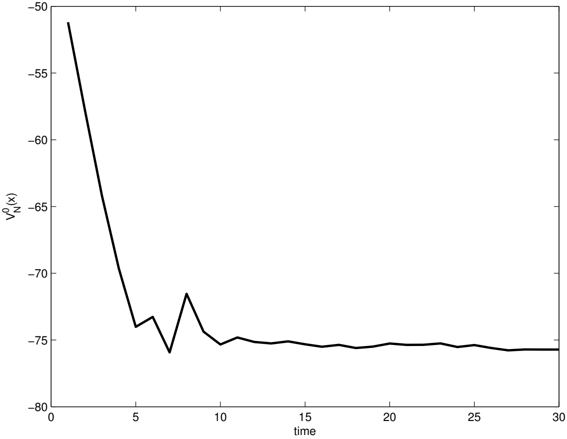

The cost is not monotonically decreasing

Assumption (Standard MPC cost function)

$$\ell_t(x,u) \geq \ell_t(x_s^*, u_s^*) = 0$$

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- $\ell(x,u)$ may be negative at some point.
- It may be $\ell(x_s^*, u_s^*) \neq 0$ or even $\ell(x_s^*, u_s^*) > \ell(x,u)$.
- This assumption does not hold anymore: MPC cost not monotonically decreasing.

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To prove stability, we need something more.

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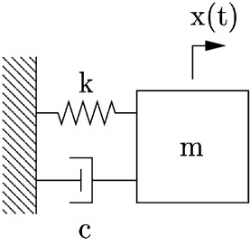
- Consider the mass-spring-damper system



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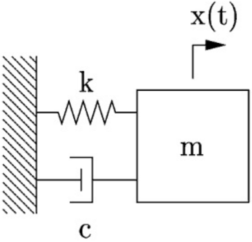

- Dynamic equation

$$m\ddot{x} + c\dot{x} + kx = 0$$

Let's go back to Lyapunov...

- Total mechanical energy = kinetic energy + potential energy

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (kx)dx = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

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- Asymptotic stability implies that energy converges to zero.
- Instability is related to the growth of mechanical energy.

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- Instability is related to the growth of mechanical energy.
- Rate of energy during system's motion:

$$\begin{aligned}\dot{V}(x) &= m\ddot{x}\dot{x} + kx\dot{x} = (-c\dot{x})\dot{x} \\ &= -c\dot{x}^2\end{aligned}$$

implies that the energy of the system is continuously dissipated by the damper until the mass settles down $(\dot{x} = 0)$.

Dissipativity

Definition (Dissipativity (Angeli et al., 2012))

A control system $f(x,u)$ is dissipative with respect to a supply rate $s : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ if there exists a function $\lambda : \mathcal{X} \to \mathbb{R}$ such that

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u)$$

for all $(x,u) \in \mathcal{X} \times \mathcal{U}$. If in addition $\rho : \mathcal{X} \to \mathbb{R}_{\geq 0}$ positive definite exists such that

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then the system is said to be strictly dissipative.

- Physical interpretation: the stored energy at time $k+1$, $\lambda(f(x,u))$ is at most equal to the sum of the energy stored at time k, $\lambda(x)$ plus the energy supplied externally $s(x,u)$.

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- Internal creation of energy is not possible: only dissipation.

Dissipativity

Given $s(x,u) = \ell(x,u) - \ell(x_s^*, u_s^*)$, then

Definition (Dissipativity (Angeli et al., 2012))

There exists a function $\lambda : \mathcal{X} \to \mathbb{R}$ *such that*

$$\begin{aligned} \min_{x,u} \quad & \ell(x,u) + \lambda(x) - \lambda(f(x,u)) \geq \ell(x_s^*, u_s^*) \\ s.t. \quad & x \in \mathcal{X}, u \in \mathcal{U} \end{aligned}$$

Defining a *rotated* stage cost function as:

$$L(x,u) = \ell(x,u) + \lambda(x) - \lambda(f(x,u))$$

we notice that

$$\min_{x \in \mathcal{X}, u \in \mathcal{U}} L(x,u) \geq \ell(x_s^*, u_s^*) = L(x_s^*, u_s^*)$$

which is what we were looking for.

Stability proof

Definition (EMPC auxiliary cost function)

$$\tilde{V}_N(x, \mathbf{u}) = \sum_{j=0}^{N-1} L(x(j), u(j))$$

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- Different cost function but same set of constraints (same $\mathcal{X}_N$).

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- Different cost function but same set of constraints (same $\mathcal{X}_N$).
- *Lemma*: the optimal solution of the auxiliary problem $\mathbf{u}^0(x)$ is identical to the original.

Stability proof

Proof:

$$
\begin{aligned}
\tilde{V}_N(x,\mathbf{u}) &= \sum_{j=0}^{N-1} L(x(j),u(j)) \\
&= \sum_{j=0}^{N-1} \ell(x(j),u(j)) + \lambda(x(j)) - \lambda(f(x(j),u(j)))
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&= \lambda(x(0)) - \lambda(x(1)) + \lambda(x(1)) - \lambda(x(2)) + \ldots \\
&\quad \ldots + \lambda(x(N-1)) - \lambda(x(N)) + \sum_{j=0}^{N-1} \ell(x(j),u(j))
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&= \lambda(x) - \lambda(x_s^*) + V_N(x,\mathbf{u})
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- $\lambda(x) - \lambda(x_s^*)$ is a constant term.
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The optimal solution is identical.

Stability proof

Assumption

1. *$f(.,.)$ and $\ell(.,.)$ are continuous. Let $\mathcal{X}_N$ be the admissible set of initial state for which the EMPC problem has a solution. Let x_s^* be in the interior of $\mathcal{X}_N$.*
2. *(Weak controllability): there exists a $\mathcal{K}_\infty$ function γ s. t. for all $x \in \mathcal{X}_N$ there exists a feasible $\mathbf{u}$ s.t.*

$$\sum_{j=0}^{N-1} \|u(j) - u_s^*\| \leq \gamma(\|x - x_s^*\|)$$

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Theorem

If $f(x,u)$ *is strictly dissipative w.r.t.* $s(x,u) = \ell(x,u) - \ell(x_s^*, u_s^*)$, *then* $\tilde{V}_N^0(x)$ *serves as a Lyapunov function of the closed-loop system and* x_s^* *is A-stable equilibrium.*

Stability proof

Proof:

- By definition

$$\min_{x \in \mathcal{X}, u \in \mathcal{U}} L(x,u) \geq \ell(x_s^*, u_s^*) = L(x_s^*, u_s^*)$$

and taking $\ell(x_s^*, u_s^*) = 0$, then there exists $\alpha_1 \in \mathcal{K}_\infty$ s.t.

$$\tilde{V}_N^0(x) \geq \alpha_1(|x - x_s^*|)$$

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- $\tilde{V}_N^0(x)$ is continuous, positive definite w.r.t. x_s^*, then there exists $\alpha_2 \in \mathcal{K}_\infty$ s.t.

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- Following standard MPC arguments (Rawlings & Mayne, 2009)

$$\begin{aligned}\tilde{V}_N^0(x^+) - \tilde{V}_N^0(x) &\leq -L(x,\kappa_N(x)) + L(x_s^*,u_s^*)\\ &\leq -\rho(x)\end{aligned}$$

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$$\min_{x\in\mathcal{X},u\in\mathcal{U}} L(x,u) \geq \ell(x_s^*,u_s^*) = L(x_s^*,u_s^*)$$

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$$\tilde{V}_N^0(x) \geq \alpha_1(|x - x_s^*|)$$

- $\tilde{V}_N^0(x)$ is continuous, positive definite w.r.t. x_s^*, then there exists $\alpha_2 \in \mathcal{K}_\infty$ s.t.

$$\tilde{V}_N^0(x) \leq \alpha_2(|x - x_s^*|)$$

- Following standard MPC arguments (Rawlings & Mayne, 2009)

$$\begin{aligned}\tilde{V}_N^0(x^+) - \tilde{V}_N^0(x) &\leq -L(x,\kappa_N(x)) + L(x_s^*,u_s^*)\\ &\leq -\rho(x)\end{aligned}$$

Then x_s^* is A-stable for the closed-loop system, with region of attraction $\mathcal{X}_N$.

Stability proof

Proof:

- To show it, as usual consider that $\tilde{V}_N^0(x^+) \leq \tilde{V}_N(x^+, \tilde{\mathbf{u}})$, where $\tilde{\mathbf{u}}$ is a feasible control sequence given by

$$\tilde{\mathbf{u}} = \{u^0(1;x), u^0(2;x), ..., u^0(N-1;x), u_s^*\}$$

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- Then

$$\begin{aligned} \tilde{V}_N^0(x^+) \leq \tilde{V}_N(x^+, \tilde{\mathbf{u}}) &= \tilde{V}_N^0(x^+) - L(x, u^0(0;x)) + L(x_s^*, u_s^*) \\ &= -\ell(x, u^0(0;x)) - \lambda(x) + \lambda(x^+) + \ell(x_s^*, u_s^*) \\ &\leq -\rho(x) \end{aligned}$$

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$$\begin{aligned} \tilde{V}_N^0(x^+) \leq \tilde{V}_N(x^+, \tilde{\mathbf{u}}) &= \tilde{V}_N^0(x^+) - L(x, u^0(0;x)) + L(x_s^*, u_s^*) \\ &= -\ell(x, u^0(0;x)) - \lambda(x) + \lambda(x^+) + \ell(x_s^*, u_s^*) \\ &\leq -\rho(x) \end{aligned}$$

Strong duality

A particular case:

Assumption (Strong duality (Diehl et al., 2011))

Let $L(x,u)$ be the rotated stage cost function given by

$$L(x,u) = \ell(x,u) + \lambda'(x - f(x,u)) - \ell(x_s^*, u_s^*)$$

where λ is a multiplier that ensures that the rotated cost exhibits a unique minimum at (x_s^*, u_s^*) for all $x \in \mathcal{X}$, $u \in \mathcal{U}$. Then there exist two $\mathcal{K}$-functions α and such that $L(x,u) \geq \alpha(|x - x_s^*|)$.

- Difficult to verify in general, but...

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- Difficult to verify in general, but...
- ... always fulfilled for linear systems and convex functions.

Strong duality: example

- Consider this linear system and quadratic cost function

$$x^+ = 0.5x + 0.5u, \quad \ell(x,u) = \|x - x_{sp}\|_Q^2 + \|u - u_{sp}\|_R^2$$

with $Q = R = 2$, and $\mathcal{U} = \{-1 \leq u \leq 1\}$

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- The optimal steady state is given by

$$\begin{aligned}(x_s, u_s) \quad &= \quad \arg\min_{x,u} \ell(x,u) \\ &\quad s.t. \quad u \in \mathcal{U}, \quad x = f(x,u)\end{aligned}$$

with $(u_s, x_s) = (1, 1)$ and $\ell(x_s, u_s) = 5$.

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with $(u_s, x_s) = (1, 1)$ and $\ell(x_s, u_s) = 5$.

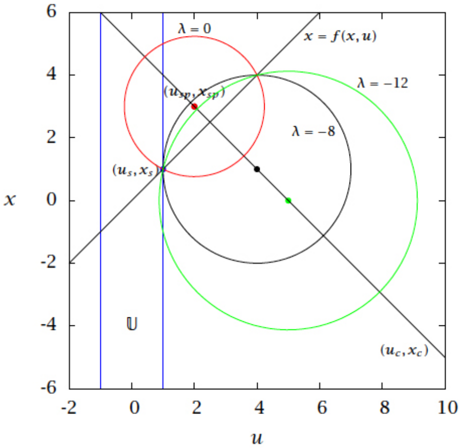
- Define the rotate cost function

$$L(x,u) = \ell(x,u) + \lambda'(x - (0.5x + 0.5u)) - \ell(x_s, u_s)$$

Strong duality: example

- Contour $L(x,u) = 0$ for $\lambda = 0$, $\lambda = -8$, $\lambda = -12$.

  - Interior: (x,u) for which $L(x,u) < 0$.
  - Exterior: (x,u) for which $L(x,u) > 0$.
  - $\lambda = 0$ (red): we have the original $\ell(x,u)$ shifted by $\ell(u_s, x_s)$.
  - The interior of the circle intersects the feasible region.



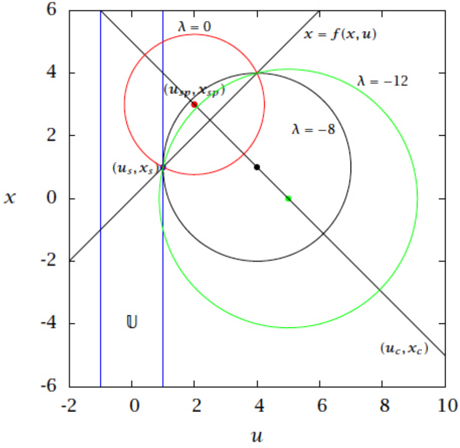
- This means that the rotated cost can be negative for a feasible input value.

Strong duality: example

- Contour $L(x, u) = 0$ for $\lambda = 0$, $\lambda = -8$, $\lambda = -12$.
  - Interior: $L(x, u) < 0$.
  - Exterior: $L(x, u) > 0$.
  - $\lambda = -8$ (black): rotation about (u_s, x_s).
  - Circle enlarged: the center moves to

$$(u_c, x_c) = (u_{sp}, x_{sp}) - \frac{\lambda}{4}(-1, 1)$$

  - Circle tangent to the feasible region.



- Therefore the entire feasible region has nonnegative rotated stage cost $L(x, u) \geq 0$

Strong duality: example

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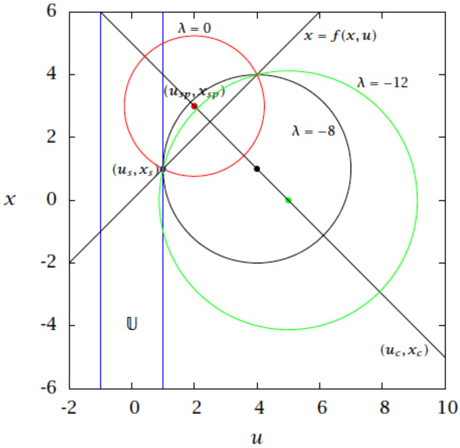
  - $\lambda = -12$ (green): rotation about (u_s, x_s).
  - Circle enlarged.
  - Circle intersects the feasible region.
  - $L(x,u) < 0$ for some feasible (u,x).



- We have only one λ for which $L(x,u) \geq 0$ for all feasible u.

Terminal cost and terminal set

Economic MPC with terminal equality constraint

Definition (EMPC cost function)

$$V_N(x, \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j))$$

Economic MPC with terminal equality constraint

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$$\begin{aligned} \min_{\mathbf{u}} \quad & V_N(x, \mathbf{u}) \\ s.t. \quad & x(0) = x,\ x(j+1) = f(x(j), u(j)) \\ & x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \\ & x(N) = x_s^* \end{aligned}$$

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- The terminal state is the optimal steady state.

EMPC with terminal inequality constraint

Definition (EMPC cost function)

$$V_N(x, \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N))$$

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- The terminal state lies in the terminal region the $\mathbb{X}_f$.

Terminal cost and terminal region

Assumption (Basic Stability Assumption)

There exists a compact terminal region $\mathbb{X}_f \subseteq \mathbb{X}$, containing x_s^* in its interior, and a control law $\kappa_f(x) : \mathbb{X}_f \to \mathbb{U}$, such that:

$$V_f(f(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x)) + \ell(x_s^*, u_s^*) \;\; \forall x \in \mathbb{X}_f$$

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Remark

This assumption implicitly requires that if $x \in \mathbb{X}_f$*, then* $f(x, \kappa_f(x)) \in \mathbb{X}_f$*, i.e.* $\mathbb{X}_f$ *is invariant.*

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Notice that $V_f(x)$ is not necessarily positive definite w.r.t x_s^*.

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Remark

Notice that $V_f(x)$ is not necessarily positive definite w.r.t x_s^*.

In what follows, it is assumed $V_f(x_s^*) = 0$.

Can we prove stability?

- As we need to define a rotated stage cost:

$$L(x,u) = \ell(x,u) + \lambda(x) - \lambda(f(x,u)) - \ell(x_s^*, u_s^*)$$

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- Now we need to define a rotated terminal cost:

$$\bar{V}_f(x) = V_f(x) + \lambda(x) - V_f(x_s^*) - \lambda(x_s^*)$$

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- Now we need to define a rotated terminal cost:

$$\bar{V}_f(x) = V_f(x) + \lambda(x) - V_f(x_s^*) - \lambda(x_s^*)$$

- So we can define the auxiliary cost function:

$$\bar{V}_N(x,\mathbf{u}) = \sum_{j=0}^{N-1} L(x(j),u(j)) + \bar{V}_f(x(N))$$

Can we prove stability?

1. Let's compare the cost functions:

Definition (EMPC optimization problem)

$$\begin{aligned} \min_{\mathbf{u}} \quad & \color{red}{V_N(x, \mathbf{u})} \\ s.t. \quad & x(0) = x, \\ & x(j+1) = f(x(j), u(j)) \\ & x(j) \in \mathcal{X}, \\ & u(j) \in \mathcal{U}, \\ & x(N) \in \mathbb{X}_f \end{aligned}$$

Definition (EMPC auxiliary optimization problem)

$$\begin{aligned} \min_{\mathbf{u}} \quad & \color{blue}{\bar{V}_N(x, \mathbf{u})} \\ s.t. \quad & x(0) = x, \\ & x(j+1) = f(x(j), u(j)) \\ & x(j) \in \mathcal{X}, \\ & u(j) \in \mathcal{U}, \\ & x(N) \in \mathbb{X}_f \end{aligned}$$

Deliver the same optimal solution?

Proof:

$$
\begin{aligned}
\bar{V}_N(x,\mathbf{u}) &= \sum_{j=0}^{N-1} L(x,u) + \bar{V}_f(x(N)) \\
&= \sum_{j=0}^{N-1} \ell(x(j),u(j)) + V_f(x(N)) - \lambda(x_s^*) - V_f(x_s^*) \\
&\quad + \sum_{j=0}^{N-1} \Big(\lambda(x(j)) - \lambda(x(j+1))) - \ell(x_s^*,u_s^*)\Big) + \lambda(x(N)) \\
&= V_N(x,\mathbf{u}) - \lambda(x_s^*) - V_f(x_s^*) - N\ell(x_s^*,u_s^*) + \lambda(x) - \lambda(x(N)) + \lambda(x(N)) \\
&= V_N(x,\mathbf{u}) - N\ell(x_s^*,u_s^*) + \lambda(x) - \lambda(x_s^*) - V_f(x_s^*)
\end{aligned}
$$

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&= V_N(x,\mathbf{u}) - N\ell(x_s^*,u_s^*) + \lambda(x) - \lambda(x_s^*) - V_f(x_s^*)
\end{aligned}
$$

- Since $N\ell(x_s^*,u_s^*)$, $\lambda(x)$, $\lambda(x_s^*)$, and $V_f(x_s^*)$ are independent of $\mathbf{u}$ for a given initial state x, then

$\bar{V}_N(x,\mathbf{u})$ and $V_N(x,\mathbf{u})$ only differ by a constant

Proof:

$$
\begin{aligned}
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&= \sum_{j=0}^{N-1} \ell(x(j),u(j)) + V_f(x(N)) - \lambda(x_s^*) - V_f(x_s^*) \\
&\quad + \sum_{j=0}^{N-1} \Big(\lambda(x(j)) - \lambda(x(j+1))) - \ell(x_s^*,u_s^*) \Big) + \lambda(x(N)) \\
&= V_N(x,\mathbf{u}) - \lambda(x_s^*) - V_f(x_s^*) - N\ell(x_s^*,u_s^*) + \lambda(x) - \lambda(x(N)) + \lambda(x(N)) \\
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$\bar{V}_N(x,\mathbf{u})$ and $V_N(x,\mathbf{u})$ only differ by a constant

- The two optimization problems deliver the same solution.

Stability assumption on the terminal cost

1. Recall the basic stability assumption

Assumption (Basic Stability Assumption)

$$V_f(f(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x)) + \ell(x_s^*, u_s^*) \;\; \forall x \in \mathbb{X}_f$$

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Lemma (Modified terminal cost)

The pair $(\bar{V}_f(.), L(.))$ satisfies the following property iff the pair $(V_f(.), \ell(.))$ satisfies the Basic Stability Assumption

$$\bar{V}_f(f(x,\kappa_f(x))) - \bar{V}_f(x) \leq -L(x,\kappa_f(x)), \quad \forall x \in \mathbb{X}_f$$

Proof:

- Take

$$V_f(f(x,\kappa_f(x))) \leq V_f(x) - \ell(x,\kappa_f(x)) + \ell(x_s^*, u_s^*)$$

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- Adding $\lambda(f(x, \kappa_f(x))) + \lambda(x)$ to both sides of the inequality

$$\begin{aligned} V_f(f(x, \kappa_f(x))) + \lambda(f(x, \kappa_f(x))) + \lambda(x) \leq\ & V_f(x) - \ell(x, \kappa_f(x)) + \ell(x_s^*, u_s^*) \\ & + \lambda(f(x, \kappa_f(x))) + \lambda(x) \end{aligned}$$

Proof:

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- Adding $\lambda(f(x,\kappa_f(x))) + \lambda(x)$ to both sides of the inequality
$$\begin{aligned} V_f(f(x,\kappa_f(x))) + \lambda(f(x,\kappa_f(x))) + \lambda(x) \;\leq\; & V_f(x) - \ell(x,\kappa_f(x)) + \ell(x_s^*, u_s^*) \\ & + \lambda(f(x,\kappa_f(x))) + \lambda(x) \end{aligned}$$
- Then
$$\begin{aligned} V_f(f(x,\kappa_f(x))) + \lambda(f(x,\kappa_f(x))) - V_f(x) - \lambda(x) \;\leq\; & -\ell(x,\kappa_f(x)) + \ell(x_s^*, u_s^*) \\ & -\lambda(x) + \lambda(f(x,\kappa_f(x))) \end{aligned}$$

Proof:

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- Then

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- Taking into account that $\bar{V}_f(x) = V_f(x) + \lambda(x) - V_f(x_s^*) - \lambda(x_s^*)$, then

$$\begin{aligned} \bar{V}_f(f(x,\kappa_f(x))) - \bar{V}_f(x) \;\le\; & -\Big(\ell(x,\kappa_f(x)) + \lambda(x) - \lambda(f(x,\kappa_f(x))) - \ell(x_s^*,u_s^*)\Big) \\ =\; & -L(x,\kappa_f(x)) \end{aligned}$$

Is $\bar{V}_f(x)$ a CLF?

- $\bar{V}_f(x)$ is continuous in $\mathbb{X}_f$ and $\bar{V}_f(x_s^*) = 0$.

Is $\bar{V}_f(x)$ a CLF?

- $\bar{V}_f(x)$ is continuous in $\mathbb{X}_f$ and $\bar{V}_f(x_s^*) = 0$.
- We proved that

$$\bar{V}_f(f(x,\kappa_f(x))) - \bar{V}_f(x) \;\leq\; -L(x,\kappa_f(x))$$

Is $\bar{V}_f(x)$ a CLF?

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- Since $L(x, u) > 0$ for $x \neq x_s^*$, then $\bar{V}_f(x) > 0$ for $x \in \mathbb{X}_f$ and $x \neq x_s^*$.

Is $\bar{V}_f(x)$ a CLF?

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- So we are OK

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  Using the standard MPC arguments, comparing $\bar{V}_N^0(x^+)$ and $\bar{V}_N^0(x)$ we get

$$\begin{aligned}\bar{V}_N^0(x^+) - \bar{V}_N^0(x) &\leq -L(x, u^0(x)) \\ &\leq -\gamma_3(|x - x_s^*|)\end{aligned}$$

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- Then $\bar{V}_N^0(x)$ is a Lyapunov function and x_s^* is an A-stable equilibrium point.

A candidate terminal cost: preliminaries

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Assumption

Functions $f(x,u)$ and $\ell(x,u)$ are twice continuously differentiable in $\mathbb{X} \times \mathbb{U}$. The linearized system $x^+ = Ax + Bu$, with $A = \left.\frac{\partial f}{\partial x}\right|_{(0,0)}$ and $B = \left.\frac{\partial f}{\partial u}\right|_{(0,0)}$ is stabilizable.

A candidate terminal set

- Choose any $u = Kx$ such that $A_K = A + BK$ is stable.

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Lemma ((Rawlings & Mayne, 2009), pp. 136-137)

There exist matrices $P > 0$, $Q > 0$ and a scalar $b > 0$ s.t., for all $\bar{b} \leq b$

$$V(f(x, Kx)) - V(x) \leq -\frac{1}{2}x'Qx, \quad \forall x \in lev_{\bar{b}}V$$

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- This implies that there exists a family of ellipsoidal control invariant sets for $f(.)$ under the control law $u = Kx$, which is given by the any level set of $V(x)$ for all $\bar{b} \leq b$

$$\mathbb{X}_f = \{x \in \mathbb{X} \mid V(x) = \frac{1}{2}x'Px \leq \bar{b}\}$$

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Lemma

Let $V_f(x) = \frac{1}{2}x'Px + p'x$, where $P > 0$ is the solution of the Lyapunov equation $A_K'PA_K - P = -Q$, for a given $Q > 0$, and $p' = q'(I - A_K)^{-1}$, for a given q. Then there exist a proper choice of Q and q such that

$$V_f(f(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x)) + \ell(x_s^*, u_s^*)$$

with $\kappa_f(x) = Kx$.

Proof:

- Recalling that we have assumed $(x_s^*, u_s^*) = (0, 0)$.

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$$\ell(x, \kappa_f(x)) \quad = \quad \ell(x, Kx) = \ell(0,0) + \nabla \ell_{(0,0)} x + \frac{1}{2} x' H_\ell(x) x$$

where $\nabla \ell_{(0,0)}$ is the gradient of $\ell(x, Kx)$ w.r.t. x evaluated in $(0,0)$, and $H_\ell(x)$ is the Hessian matrix of $\ell(x, Kx)$ w.r.t. x.

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- Define $q = \nabla \ell_{(0,0)}$. Moreover, let

$$\lambda^* = \max_{x \in \mathcal{C}} \{\lambda_M(H_\ell(x))\}$$

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- Therefore

$$\begin{aligned} \ell(x, Kx) - \ell(0,0) \quad &= \quad \nabla \ell_{(0,0)} x + \frac{1}{2} x' H_\ell(x) x \\ &\leq \quad q' x + \frac{1}{2} x' Q x \end{aligned}$$

Proof:

- Take the increment of the terminal cost under $\kappa_f(x) = Kx$, with $f(x, Kx) = A_K x$

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V_f(f(x, Kx)) - V_f(x) &= \frac{1}{2}x'A_K'PA_Kx + p'A_Kx - \frac{1}{2}x'Px - p'x \\
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V_f(f(x,Kx)) - V_f(x) &= -\frac{1}{2}x'Qx - q'x \\
&\leq -\ell(x,Kx) + \ell(0,0) = -\ell(x,Kx) + \ell(x_s^*, u_s^*)
\end{aligned}
$$

- Hence, this choice of $V_f(x)$ satisfies the basic stability assumption in a compact set $\mathbb{X}_f$ given by $\text{lev}_{\bar{b}}V$, where $V(x) = \frac{1}{2}x'Px$.

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Theorem (Average asymptotic performance (Angeli et al., 2012))

Let $x(0) \in \mathcal{X}_N$ a feasible initial condition. Then the asymptotic average performance of the closed-loop system is not worse then the one of the best admissible steady state (x_s^*, u_s^*).

- Only asymptotically... not in the transient.

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Proof:

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$$\begin{aligned}\liminf_{T\to\infty} \frac{\sum_{k=0}^{T} V_N^0(x(k+1)) - V_N^0(x(k))}{T+1} &\leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x_s^*, u_s^*) - \ell(x(k), u(k))}{T+1} \\ &= \ell(x_s^*, u_s^*) - \limsup_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1}\end{aligned}$$

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$$V_N^0(x^+) - V_N^0(x) \leq \ell(x_s^*, u_s^*) - \ell(x, u)$$

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$$\begin{aligned}\liminf_{T\to\infty} \frac{\sum_{k=0}^{T} V_N^0(x(k+1)) - V_N^0(x(k))}{T+1} &\leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x_s^*, u_s^*) - \ell(x(k), u(k))}{T+1} \\ &= \ell(x_s^*, u_s^*) - \limsup_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1}\end{aligned}$$

- Assuming $\ell(x, u) \geq 0$ (without loss of generality)

$$\begin{aligned}\liminf_{T\to\infty} \frac{\sum_{k=0}^{T} V_N^0(x(k+1)) - V_N^0(x(k))}{T+1} &= \liminf_{T\to\infty} \frac{V_N^0(x(T+1)) - V_N^0(x(0))}{T+1} \\ &\geq \liminf_{T\to\infty} \frac{-V_N^0(x(0))}{T+1} = 0\end{aligned}$$

Average asymptotic performance

Proof:

- Following standard MPC arguments:

$$V_N^0(x^+) - V_N^0(x) \leq \ell(x_s^*, u_s^*) - \ell(x, u)$$

- Take asymptotic averages

$$\begin{aligned}
\liminf_{T\to\infty} \frac{\sum_{k=0}^{T} V_N^0(x(k+1)) - V_N^0(x(k))}{T+1} &\leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x_s^*, u_s^*) - \ell(x(k), u(k))}{T+1} \\
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- Then

$$\limsup_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1} \leq \ell(x_s^*, u_s^*)$$

Example: yet another CSTR

CSTR with parallel reactions (Müller et al., 2014a).

$$\begin{aligned} R &\rightarrow P_1 \\ R &\rightarrow P_2 \end{aligned}$$

P_1 desired product, P_2 waste product.
From dimensionless mass balance:

$$\begin{aligned} \dot{x}_1 &= 1 - 10^4 x_1^2 e^{-1/x_3} - 400 x_1 e^{-0.55/x_3} - x_1 \\ \dot{x}_2 &= 10^4 x_1^2 e^{-1/x_3} - x_2 \\ \dot{x}_3 &= u - x_3 \end{aligned}$$

x_1, x_2: concentrations of R, P_1. x_3: temperature in the reactor.
u: heat flow into the reactor.
$0 \leq x_i$, $0.049 \leq u_i \leq 0.449$, $T_s = 0.1$

Example: yet another CSTR

$$\ell(x,u) = -x_2, \qquad N = 15$$

$$x_s^* = (0.0832, 0.0846, 0.1491), \quad u_s^* = (0.1491)$$

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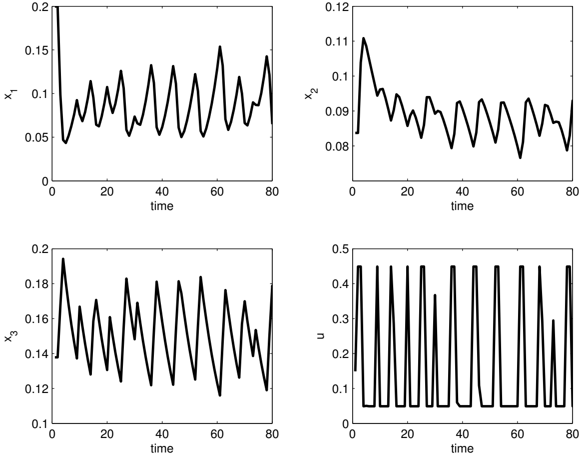

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$$\limsup_{T\to\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1} \quad \leq \quad -0.0846$$

Forcing asymptotic stability

A trick for asymptotic stability

- In general x_s^* may not be stable for an economic cost $\ell(x, u)$.

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with $\alpha(.)$ positive definite with respect to (x_s^*, u_s^*).

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- Take $\alpha(x,u) = \|x - x_s^*\|_Q^2 + \|u - u_s^*\|_R^2$.
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- If $\alpha(x,u) = 0$, $\bar{\ell}(x,u)$ is a pure economic cost.
- If $\ell(x,u) = 0$, $\bar{\ell}(x,u)$ is a pure tracking cost.

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$$\bar{\ell}_1(x,u) = \ell(x,u) + \|x - x_s^*\|_Q^2 + \|u - u_s^*\|_R^2, \qquad Q = I_3, \qquad R = 10$$

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Measure of the average profit:

$$\Phi = \frac{1}{T}\sum_{k=0}^{T} x_2(k)$$

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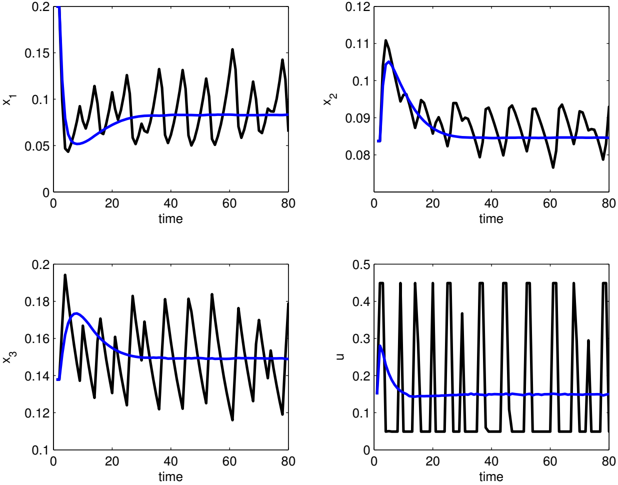

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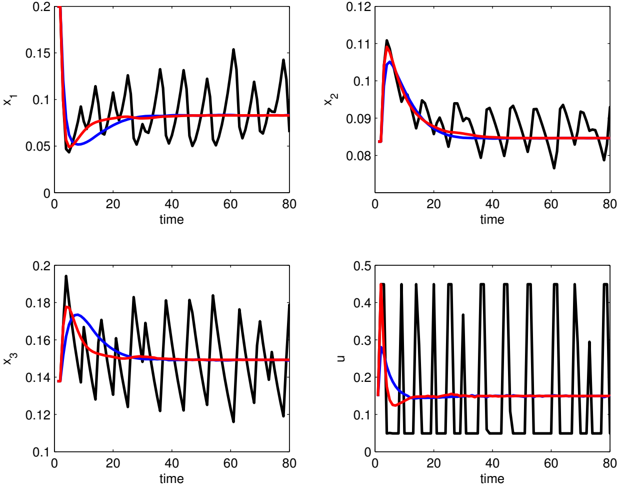

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Measure of the average profit:

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	Economic $\ell(x,u)$	Economic $\overline{\ell}(x,u)_1$	Economic $\overline{\ell}(x,u)_2$
Φ	0.0891	0.0867	0.0869

Example: CSTR

Isothermal CSTR with consecutive-competitive reactions (Rawlings et al., 2012).

$$\begin{aligned} P_0 + B &\rightarrow P_1 \\ P_1 + B &\rightarrow P_2 \end{aligned}$$

From dimensionless mass balance:

$$\begin{aligned} \dot{x}_1 &= u_1 - x_1 - \sigma_1 x_1 x_2 \\ \dot{x}_2 &= u_2 - x_2 - \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_3 &= -x_3 + \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_4 &= -x_4 + \sigma_2 x_2 x_3 \end{aligned}$$

x_1, x_2, x_3, x_4: concentrations of P_0, B, P_1, P_2. $0 \leq x_i \leq 10$
u_1, u_2: inflow rates of P_0 and B. $0 \leq u_i \leq 10$
$\sigma_1 = 1$, $\sigma_2 = 0.4$, $T_s = 0.1$

Example: CSTR

$$\ell(x,u) = -\rho_1 x_3 + \rho_2(u_1+u_2), \qquad \rho = (7,1), \qquad N = 10$$

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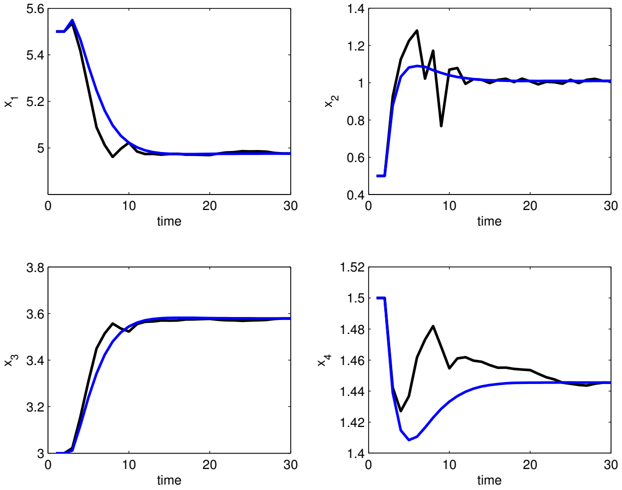

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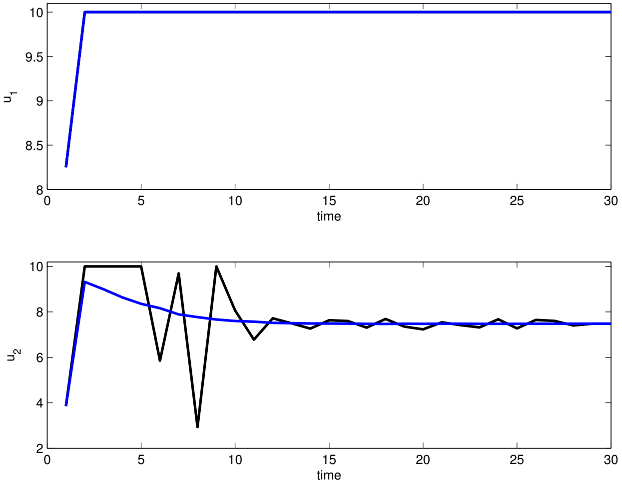

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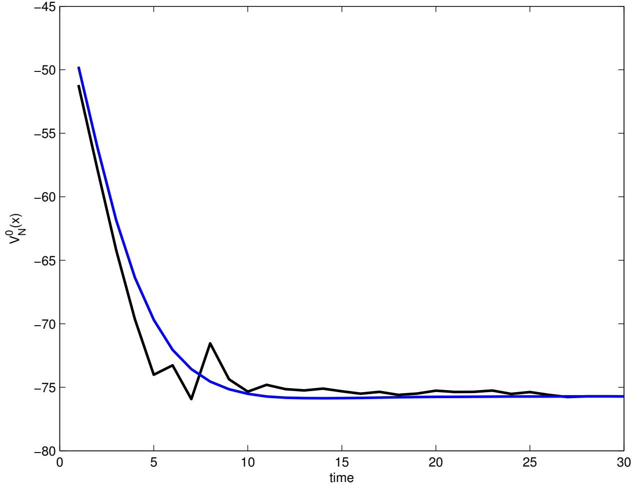

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	Economic $\ell(x,u)$	Economic $\bar{\ell}(x,u)$	Tracking
Φ	6.83	6.81	6.80
Ψ	23.29	23.68	23.90

EMPC for a changing economic criterion

What if the economic criterion changes?

- The economic cost may change due to
  - Market fluctuations: cause changes in the cost function.
  - Disturbances: cause changes in the constraints.

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Challenge: designing an economic MPC that ensures feasibility, under any change of the economic cost function.

EMPC for changing economic criterion

Definition (EMPC cost function)

$$V_N(x; \mathbf{u}, x_a, u_a) = \sum_{j=0}^{N-1} \ell(x(j) - x_a + x_s^*, u(j) - u_a + u_s^*, p) + V_O(x_a, x_s^*)$$

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Definition (EMPC optimization problem)

$$\begin{array}{cl} \min\limits_{\mathbf{u}, x_a, u_a} & V_N(x; \mathbf{u}, x_a, u_a) \\ s.t. & x(0) = x,\ x(j+1) = f(x(j), u(j)) \\ & x(j) \in \mathcal{X},\ u(j) \in \mathcal{U},\ j \in \mathbb{I}_{[0,N-1]} \\ & x(N) = x_a,\ x_a = f(x_a, u_a) \end{array}$$

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- (x_a, u_a) admissible equilibrium point.
- $V_O(x_a, x_s^*)$ a positive definite convex function such that the unique minimizer is (x_s^*, u_s^*). (i.e., the norm of a distance).

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- *Local economic optimality*: The effort to maintain feasibility may cause suboptimality during the transient phase, in the sense that the performance may differ from standard EMPC. This happens while $x(k) \in \mathcal{X}_N^t \setminus \mathcal{X}_N$.
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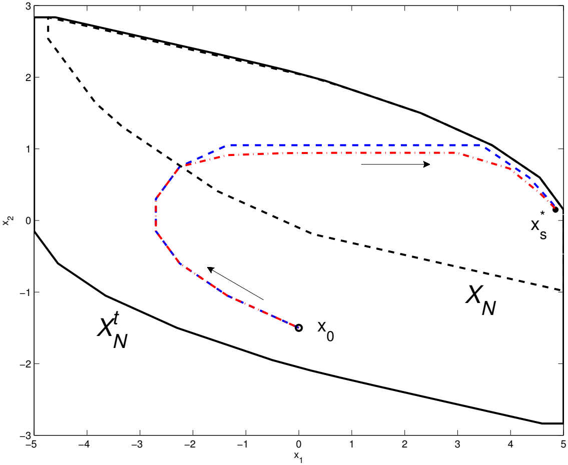
- If p converges to a constant values, asymptotic stability of (x_s^*, u_s^*) can be proved.
- The particular cost need the knowledge of (x_s^*, u_s^*).

EMPC for changing economic criterion

Properties:

- Guaranteed feasibility as MPCT, while $x(k) \in \mathcal{X}_N^t \setminus \mathcal{X}_N$.
- Economic optimality as EMPC, once $x(k) \in \mathcal{X}_N$.



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$$L(x - x_a + x_s^*, u - u_a + u_s^*, p) = \ell(x - x_a + x_s^*, u - u_a + u_s^*, p) + \lambda(x) - \lambda(x^+)$$

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- Define

$$\begin{aligned} \tilde{V}_N(x; \mathbf{u}, x_a, u_a) &= \sum_{j=0}^{N-1} \ell(x(j) - x_a + x_s^*, u(j) - u_a + u_s^*, p) + \tilde{V}_O(x_a, x_s^*) \\ \tilde{V}_O(x, x_s^*) &= V_O(x, x_s^*) + \lambda(x) - \lambda(x_s^*) \end{aligned}$$

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- $\tilde{V}_N(x; \mathbf{u}, x_a, u_a) - V_N(x; \mathbf{u}, x_a, u_a) = \lambda(x) - \lambda(x_s^*)$.

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- $\tilde{V}_N(x; \mathbf{u}, x_a, u_a) - V_N(x; \mathbf{u}, x_a, u_a) = \lambda(x) - \lambda(x_s^*)$.
- The optimal value of the auxiliary cost, $\tilde{V}_N^0(x)$ is a Lyapunov function (Ferramosca et al., 2014).

EMPC for changing economic criterion

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- Define the rotated cost

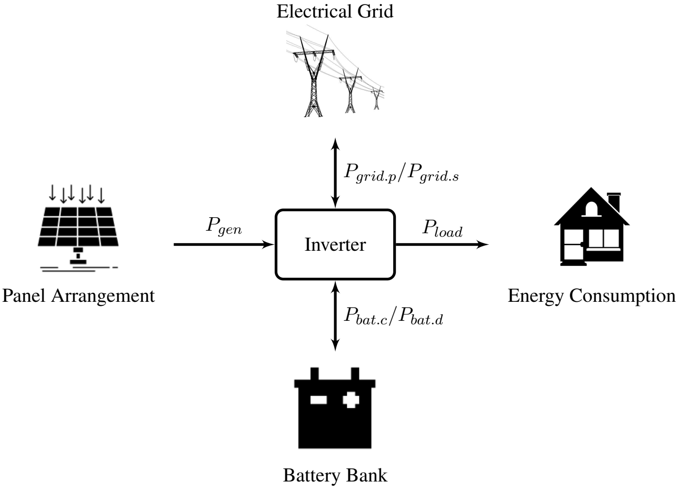
$$L(x-x_a+x_s^*,u-u_a+u_s^*,p)=\ell(x-x_a+x_s^*,u-u_a+u_s^*,p)+\lambda(x)-\lambda(x^+)$$

- Define

$$\begin{aligned}\tilde{V}_N(x;\mathbf{u},x_a,u_a) &= \sum_{j=0}^{N-1}\ell(x(j)-x_a+x_s^*,u(j)-u_a+u_s^*,p)+\tilde{V}_O(x_a,x_s^*)\\ \tilde{V}_O(x,x_s^*) &= V_O(x,x_s^*)+\lambda(x)-\lambda(x_s^*)\end{aligned}$$

- $\tilde{V}_N(x;\mathbf{u},x_a,u_a)-V_N(x;\mathbf{u},x_a,u_a)=\lambda(x)-\lambda(x_s^*)$.
- The optimal value of the auxiliary cost, $\tilde{V}_N^0(x)$ is a Lyapunov function (Ferramosca et al., 2014).
- No need to calculate λ... but x_s^* is still needed.

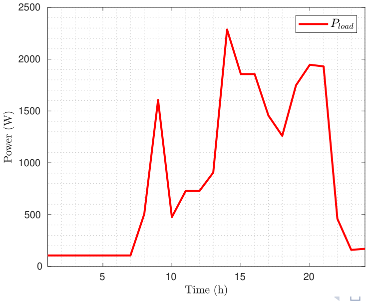
Case study: EMPC for micgrogrids managment



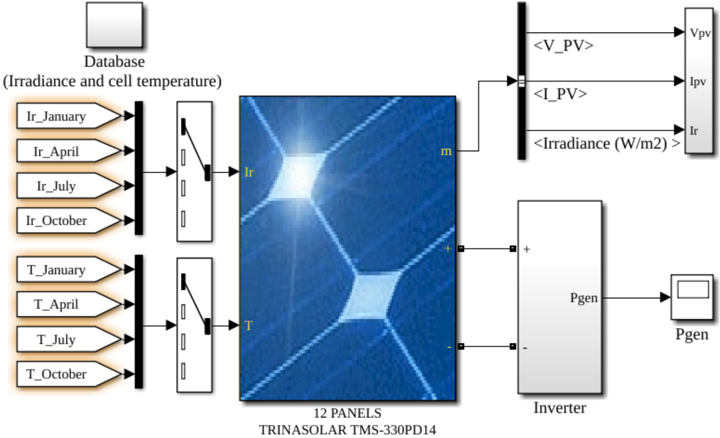
Economic goal: manage the energy exchange with the main grid and minimize the ageing cycle of the energy storage devices.

Energy consumption

Equipment	Quantity	Power in kW	Hours
9 W LED Lamp 9 W	6	0.009	6
LED TV 32 a 50"	2	0.15	2
Notebook	1	0.025	6
Refrigerator	1	0.1	24
Water heater	1	1.5	0.25
Air conditioning	1	1.4	5
Washing machine	1	0.4	1
Microwave and iron	1	3.2	0.35
Various	1	0.8	0.2



Solar panels arrangement



A Simulink simulator has been developed to obtain the generated power: P_{gen}.

Battery bank

Nominal voltage	13.2 V
Nominal capacity	400 Ah
Continuous charging current	400 A (1 C)
Continuous discharge current	500 A (1.25 C)
Useful life (cycles)	3500 (80% of DOD)
Charging efficiency ($\eta_{bat.c}$)	0.92
Discharging efficiency ($\eta_{bat.d}$)	0.90

Number of batteries	4
Nominal voltage (V_{nom})	52.8 V
Nominal capacity (C_{Ah})	400 Ah
Continuous charging current (I_{1c})	400 A (1 C)
Nominal power capacity (C_{Wh})	21120 Wh
Connection type	Serie

Autonomy: $A = \frac{C_{Wh}\ \eta_{bat.d}\ \text{dod}}{C_{daily}} = \frac{21120\ (\text{Wh})\ 0.9\ 0.6}{12570\ (\text{Wh})} = 0.907$

State space model of the microgrid

A state-space discrete-time general-linear model for microgrid, is represented by the following linear time-invariant discrete time system:

$$x(k+1) = Ax(k) + Bu(k) \tag{1a}$$

$$B_u u(k) + E_w w(k) = 0 \tag{1b}$$

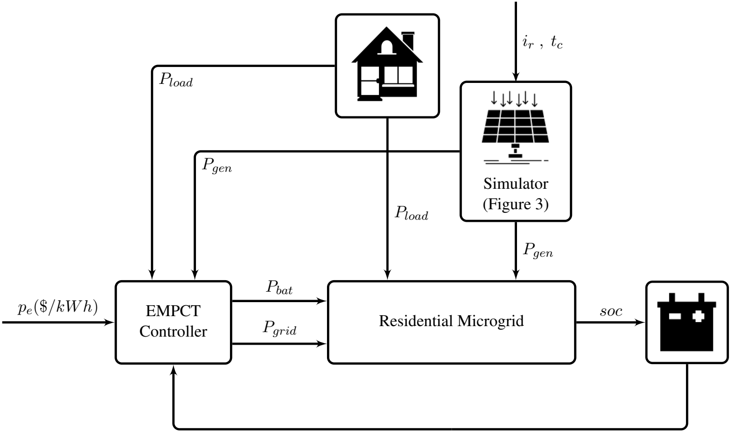
Equation (1a) describes the general dynamics of a storage system, where $x(k)$ is the state of charge of the battery; while equation (1b) corresponds to Kirchhoff's current law at the microgrid node (*power balance* of the system).

The control variable $u(k)$ and the disturbance $w(k)$ are given by:

$$u_k = \left[\begin{array}{cccc} P_{bat.c|k} & P_{bat.d|k} & P_{grid.p|k} & P_{grid.s|k} \end{array}\right]'$$

$$w_k = \left[\begin{array}{cc} P_{gen|k} & P_{load|k} \end{array}\right]'$$

EMPC for micgrogrids managment



Economic goal: manage the energy exchange with the main grid and minimize the ageing cycle of the energy storage devices.

Formulation of the EMPC

The economic controller has the purpose to fulfill the objectives listed below:

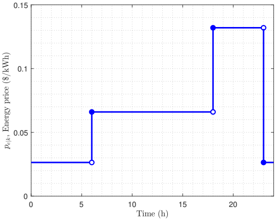
1. To fulfill the demand of users in the residence.
2. To maximize the economic benefit of energy exchange with the main electrical grid.
3. To consider the cost of use for the energy stored in the battery bank.
4. To minimize the aging cycle for lithium-ion batteries, prolonging their useful life.

Based on the above, the economic cost function ℓ_{eco} will be made up of two terms: the first of them is called ***Cost of use for the electricity grid*** and, the second, ***Cost of use for the battery bank***.

$$\ell_{eco}(x, u, p) = \ell_{grid}(u, p_e) + \ell_{bat}(x, u)$$

Cost of use for the electricity grid

A variable daily price scenario is considered $p_{e|k}$, for the purchase and sale of energy. It is composed by three time zones, called *peak*, *valley* and *rest*. *Peak* hours go from 18:00 to 22:59 h, the *valley* goes from the 23:00 to 05:59 h and the *rest* starts at 6:00 to 17:59 h.



The prices are expressed in \$/kWh. Based on the above discussion, the expression representing the cost of using the grid is:

$$\ell_{grid}(u_k, p_{e|k}) = \lambda_{grid.p}\, p_{e|k} P_{grid.p|k}\, T_s + \lambda_{grid.s}\, p_{e|k}\, P_{grid.s|k}\, T_s$$

where, $\lambda_{grid.p}$ and $\lambda_{grid.s}$, represent priority weight constants for the purchase or sale of energy and T_s is the sampling period expressed in hours.

Cost of use for the battery I

The aging of batteries is influenced by external factors (temperature, humidity, etc.) and internal factors (depth of discharge, acting currens, over charge/discharge). The following cost is proposed:

$$\ell_{bat}(x,u) = \ell_{bat_1}(u) + \ell_{bat_2}(u) + \ell_{bat_3}(x)$$

where

- $\ell_{bat_1}(u)$ is the cost of use considering the cost for a replacement: having the replacement cost C_{re} and the energy storage capacity C_{Wh} for the batteries that make up the storage system, it follows that:

$$\ell_{bat_1}(u_k) = \lambda_{bat.c}\frac{C_{re}}{C_{Wh}}P_{bat.c|k}T_s + \lambda_{bat.d}\frac{C_{re}}{C_{Wh}}P_{bat.d|k}T_s$$

  Here, as in the cost of use for the grid, λ_{bat_c} and λ_{bat_d}, represent weight constants for priority.

Cost of use for the battery II

- $\ell_{bat_2}(u)$ is the cost due to the current on the bank

$$\ell_{bat_2}(u) = \lambda_{bat.2} \frac{P_{bat.d} + P_{bat.c}}{V_{nom} I_{1c}}$$

where, again, $\lambda_{bat.2}$ is a weighting constant.

- $\ell_{bat_3}(x)$ is the evolution of the state of charge: the degradation becomes more noticeable when the variation of the soc is significant from one instant to another

$$\ell_{bat_3}(x_k) = \|x_{k+1} - x_k\|_C^2$$

EMPC Controller

The cost functional for the optimal control problem to be solved is:

$$V_N(x,p_e;\mathbf{u}) = \sum_{j=0}^{N-1} \ell_{eco}\left(x - x_a + x_s^*, u - u_a + u_s^*, p_e\right) + \|x - x_a\|_Q^2 + \|u - u_a\|_R^2 + V_O\left(x_a, x_s^*\right)$$

where

- (x_a, u_a) are the artificial variables (as in the MPCT seen in the previous lesson)
- $V_O\left(x_a.x_s^*\right)$ is the offset cost function
- $\|x - x_a\|_Q^2 + \|u - u_a\|_R^2$ is a regularization term
- (x_s^*, u_s^*) is the economically optimal steady state given by

$$\begin{aligned}(x_s^*, u_s^*) = \arg\min_{\mathbf{x},\mathbf{u}} & \ \ell_{eco}(x,u,p_e) \\ s.t. \ \ & x = Ax + Bu, \\ & x \in \mathbb{X}, \quad u \in \mathbb{U}.\end{aligned}$$

EMPC Controller

The optimal control problem is given by

$$\min_{\mathbf{u},x_a,u_a} \quad V_N(x,p_e;\mathbf{u},x_a,u_a)$$

subject to:

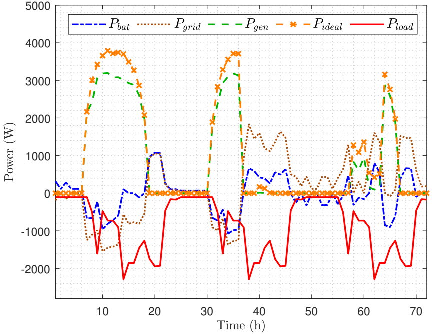
$$\begin{aligned}
& x_0 = x, \quad x_a = Ax_a + Bu_a \\
& x(j+1) = Ax(j) + Bu(j), \qquad j \in \mathbb{I}_{0:N-1} \\
& B_u u(j) + E_w w(j) = 0, \qquad j \in \mathbb{I}_{0:N-1} \\
& x(j) \in \mathbb{X}, \quad u(j) \in \mathbb{U}, \qquad j \in \mathbb{I}_{0:N-1} \\
& x(N) = x_a
\end{aligned}$$

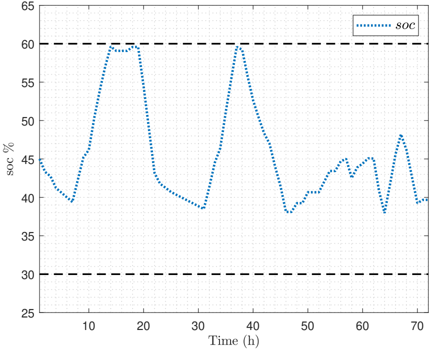
with constraint $\mathbb{X}$ given by $30\% \leq soc_k \leq 60\%$ and constraint $\mathbb{U}$ given by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} P_{bat.c|k} \\ P_{bat.d|k} \\ P_{grid.p|k} \\ P_{grid.s|k} \end{pmatrix} \leq \begin{pmatrix} 2400 \\ 2400 \\ 3000 \\ 2100 \end{pmatrix}$$

Simulation results

$N = 24$. Month of January (Summer) with changing weather conditions. Irradiance and temperature data refer to Santa Fe, Argentina.

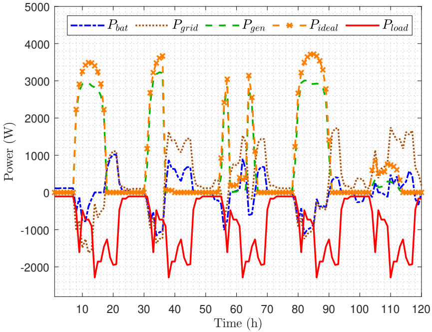


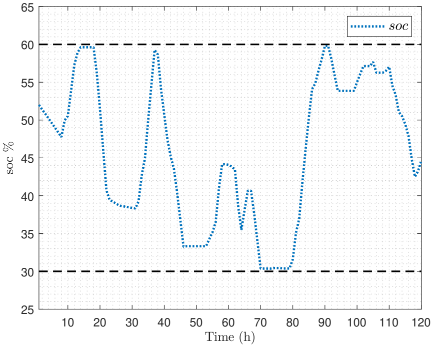


Ideal power profile (P_{ideal}) represents the product of the irradiance (W/m^2) by the square meters (m^2) of the array of panels.

Simulation results

$N = 24$. Month of April (Autumn) with changing weather conditions. Irradiance and temperature data refer to Santa Fe, Argentina.

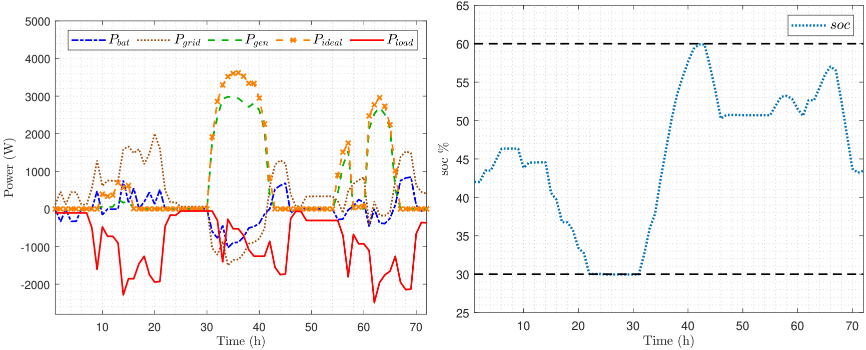




Ideal power profile (P_{ideal}) represents the product of the irradiance (W/m^2) by the square meters (m^2) of the array of panels.

Simulation results

$N = 24$. Month of October (Spring) with changing weather conditions. Irradiance and temperature data refer to Santa Fe, Argentina.



Ideal power profile (P_{ideal}) represents the product of the irradiance (W/m^2) by the square meters (m^2) of the array of panels.

Outline

Overview of recent results

MPC community very active on this topic:

- Periodic terminal constraint $x(N) = x^*(q)$, for q index of a periodic trajectory with period Q (Angeli et al., 2012).
- Economic MPC without terminal constraint (Grüne, 2013).
- Transient and asymptotic average constraint: $Av[\ell(x,u)] \subseteq (-\infty, \ell(x_s^*, u_s^*))$ and $Av[y = h(x,u)] \subseteq \mathcal{Y}$ (Angeli et al., 2012; Müller et al., 2014a,b).
- Lyapunov based Economic MPC (Heidarinejad et al., 2012).
- Necessity of dissipativity for steady state operation (Müeller et al., 2015; Müller et al., 2015)
- Economic MPC with time varying cost (Ferramosca et al., 2014; Angeli et al., 2015)
- ...and much more
- A survey: "A tutorial review of economic model predictive control methods". M. Ellis, H. Durand, and P. D. Christofides. " *Journal of Process Control*, 24(8), 2014.

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Thanks for your attention!