UNIVERSITÀ
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Dipartimento
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dell'Informazione e della Produzione

Lesson 14.

Fault diagnosis II

Model-based approaches

DATA SCIENCE AND AUTOMATION COURSE

MASTER DEGREE SMART TECHNOLOGY ENGINEERING

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Outline

1. Schematic of the approach
2. Dynamical systems
3. Parity space approach
4. Diagnostic observer
5. Application to EMA fault detection

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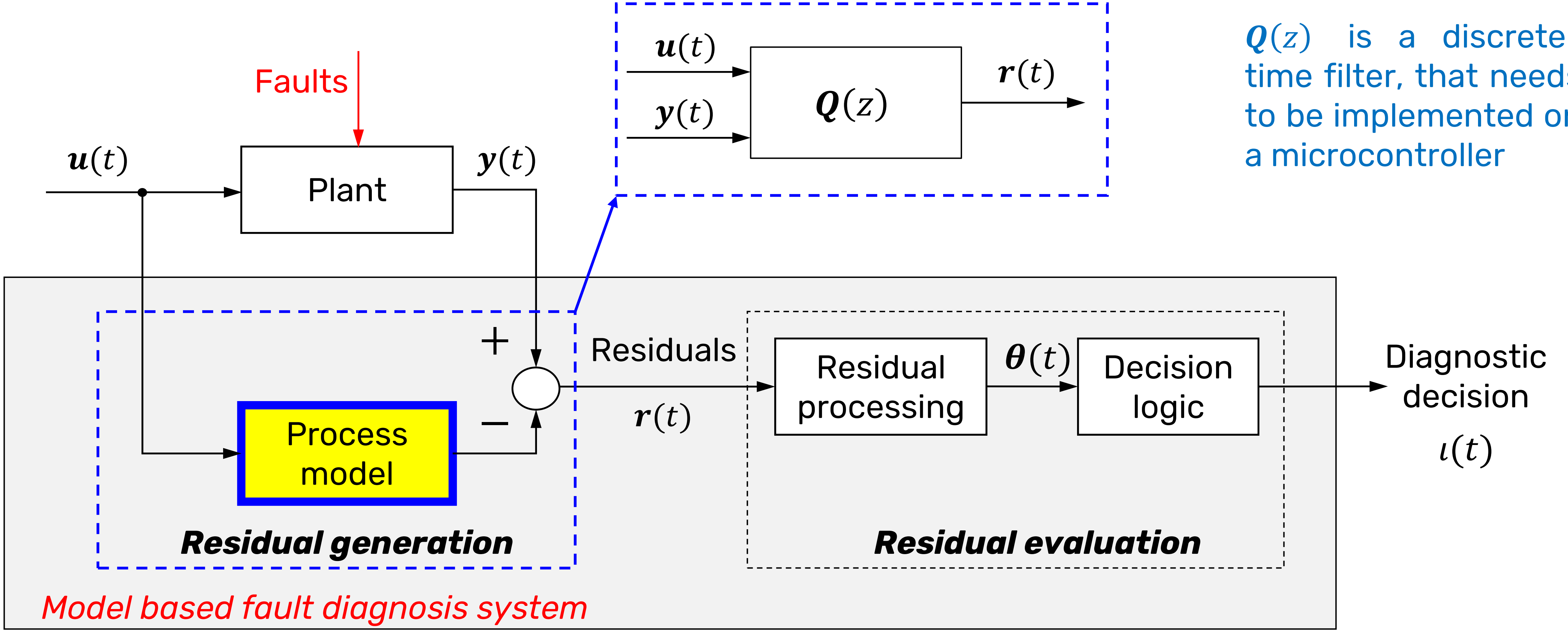
1. **Schematic of the approach**
2. Dynamical systems
3. Parity space approach
4. Diagnostic observer
5. Application to EMA fault detection

Model-based fault diagnosis

A model of the system is developed to design a **residual generator** $\boldsymbol{Q}(z)$

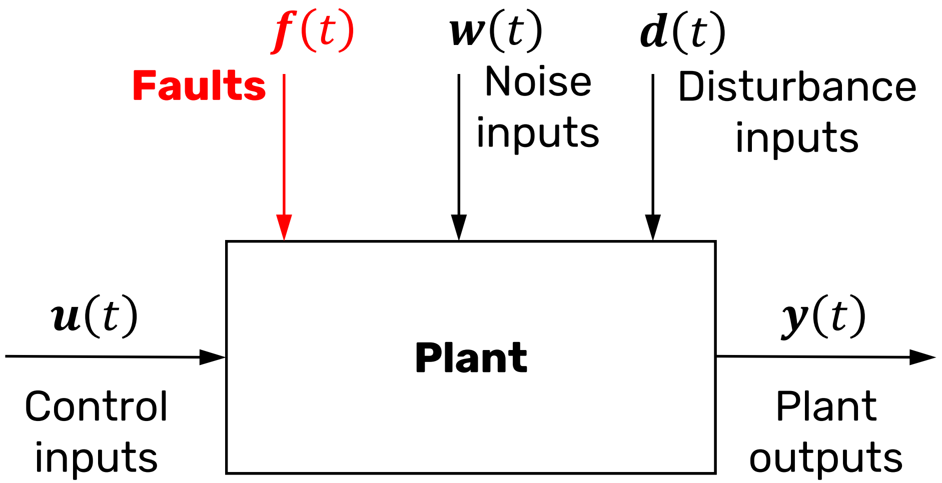


$\boldsymbol{Q}(z)$ is a discrete-time filter, that needs to be implemented on a microcontroller

Modeling a plant with faults

A **modeling scheme for fault diagnosis** considers the following external inputs:

- $\boldsymbol{y}(t)$: plant outputs (measureable)
- $\boldsymbol{u}(t)$: plant inputs (measureable)
- $\boldsymbol{d}(t)$: disturbance inputs *(not measureable)*
- $\boldsymbol{w}(t)$: noise inputs *(not measureable)*

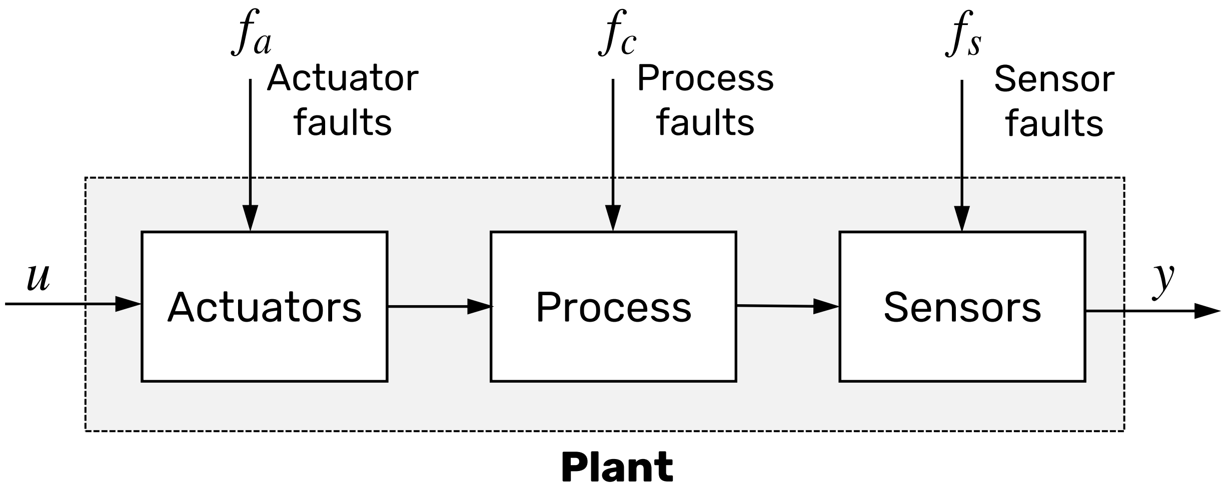


- **Disturbances:** include unknown uncontrollable inputs *(wind shears, crosswinds, load variations)*
  - ✓ If the transfer function from $\boldsymbol{d}$ to $\boldsymbol{y}$ is known, their effect on $\boldsymbol{y}$ can be **completely rejected**
- **Noises:** include uncertainties such as random noises or parametric model uncertainties
  - ✓ They can **only be attenuated** up to a certain extent

Classification of fault types

Faults can be classified based on their **location (physical classification)...**

- **Actuator faults:** a change in the characteristics of an actuator leading to a loss of efficiency or even to a complete breakdown *(jamming, runaway, floating, loss of effectiveness)*
- **Sensor faults:** erroneous measurements obtained with a defective sensor *(freezing, drift, bias, loss of accuracy)*
- **Process faults:** malfunction of an internal component due to excessive variation of some physical parameter *(structural damage, leakage, shortcut)*



Classification of fault types

…or based on how it is possible to **model their effect on the system:**

- **Additive faults:** fault modelled as a *fictive input* f, which acts independently on the plant inputs and outputs. The case $f = 0$ corresponds to *the fault-free case.* *(some types of **actuator faults** (jamming, runaway, oscillatory fault case), some types of **sensor faults** (bias, drift), operational wear and tear)*

- **Parametric faults:** fault whose effects on the plant depend on the magnitude of some internal signals or known inputs. *(**parametric faults**, some types of **actuator faults** (loss of efficiency, disconnection, stall load)*

The classification is important to adopt the **correct modeling** and **diagnosis technique**

Outline

Dynamical systems

A system (physical object that accepts inputs and produces outputs) is said to be **dynamical** if the output $y(t)$ at a certain time t **does not depend only** of the input $u(t)$ at the same time t, but also on the initial system condition $\boldsymbol{x}(0)$.

- In an **electromechanical motor**, the relation between the motor current and the motor speed can be described by a dynamical model
- The force generated by a **skeletal muscle** contraction will depend by the viscous damping given by the tissue and on the elastic storage properties by the sarcolemma and tendons
- **Flow equation** of blood through the vessels depend on pressures dynamics

An example of **static system** is the resistor: $i(t) = \Delta V(t)/R$,

$i(t) = \text{current in the resistor [A]}, \Delta V(t) = \text{voltage drop on the resistor [V]}, R = \text{resistance [}\Omega\text{]}.$

Mathematical models of dynamical systems

A **dynamical model** is **mathematical object** that can be used to analyze the behavior of a dynamical system

Tipically they are represented using a **set of equations** that explain the **relation between the variables** involved in the phenomenon, i.e. how the variables **evolve in time**

Models represent only a **simplified version** of the real phenomenon:

- «All models are wrong, but some are useful»

There are **different types** of dynamical models:

- Linear \ nonlinear
- Time invariant \ time variant
- Continuous \ discrete time

Example 1: Infectious disease

We want to model the **number of people infected** with a contagious disease

Suppose to **measure** the number of infected people one time every day **(sampling time)**

Let's define as $I(t)$ the **number of infected** people at day $t \in \mathbb{N}_{>0}$ (discrete-time case)

t is a multiple of the sampling period T_s

$$I(t+1) = I(t) + \lambda \cdot I(t)$$

Difference between new infected and deaths\recovered

- λ is the disease spreading rate (the higher λ, the faster the spreading. If negative, the epidemic subsides)

The number of infected people **at a certain day** depends on the number of infected in the **previous day** → The dynamical system has a **«memory» property**

Example 1: Simulation

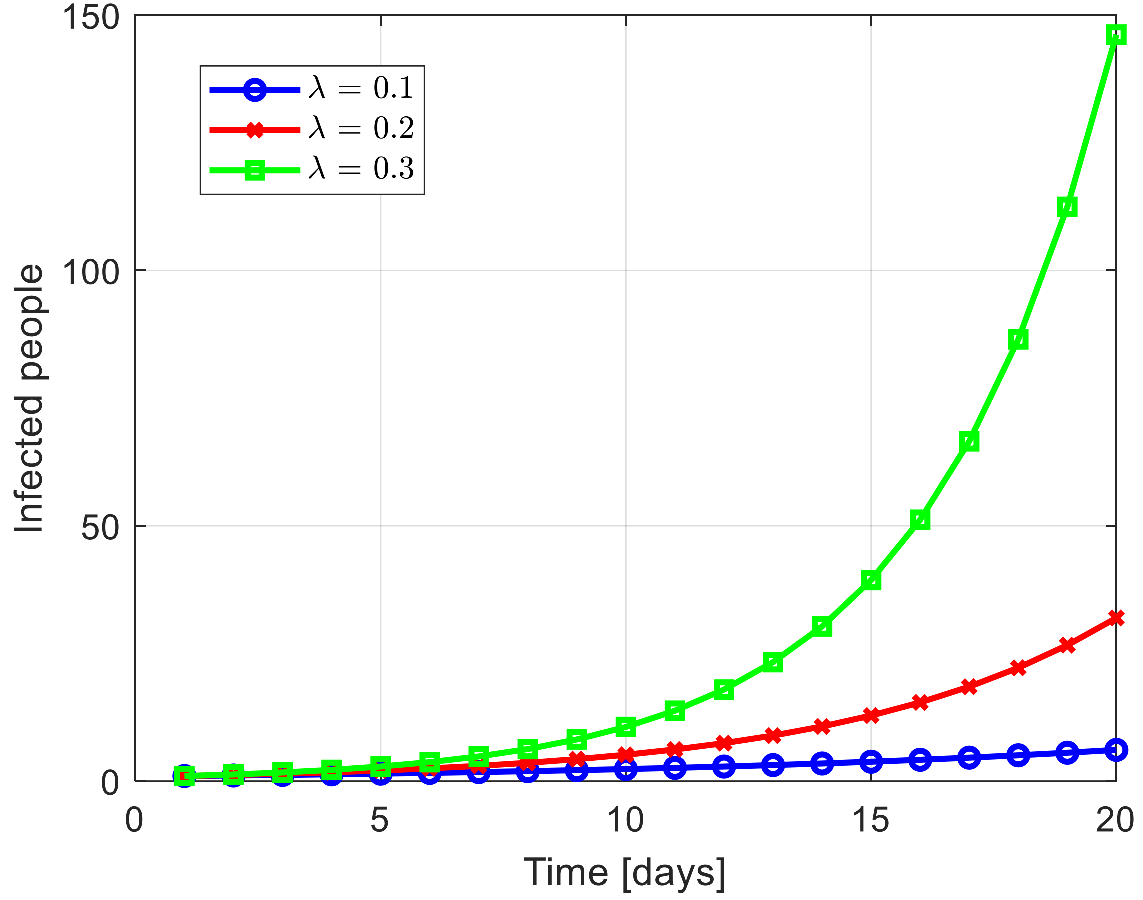
In order to analyze the phenomena, we can simulate the model. To simulate this model we have to know the **initial value** of infected people

```
I(1) = 1;        Initial condition
T = 20; lambda = 0.1;
for t = 2 : 1 : T
    I(t) = I(t-1) + lambda*I(t-1);
end
```

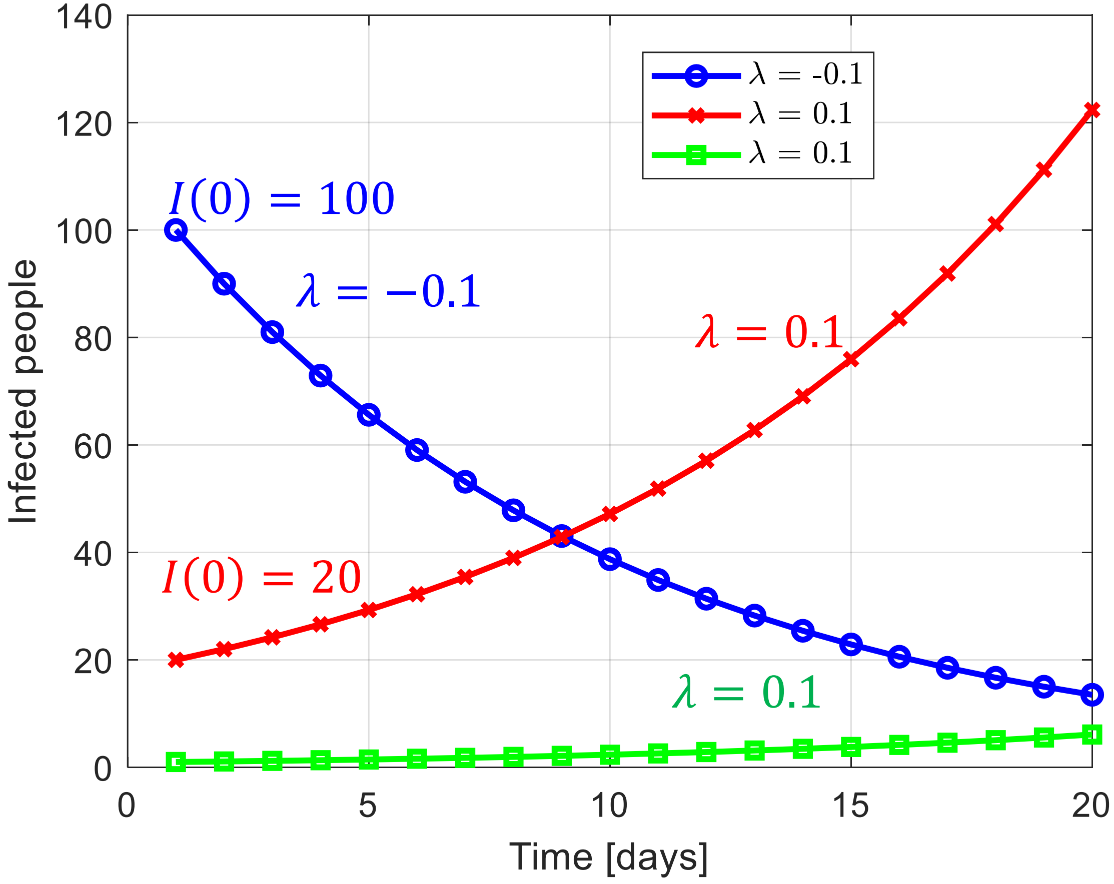
The number of infected people **at a certain day** depends on the number of infected in the **previous day** → The dynamical system has a **«memory» property**

Example 1: Simulation

Different rates λ, $I(0) = 1$



Different initial conditions $I(0)$



Example 1: Assumptions

The previous model is a **simplified** version of the phenomenon. It represents reality up to a certain extent

$$I(t+1) = I(t) + \lambda \cdot I(t)$$

A lot of **assumptions** are used, for example:

- The are **infinite** infectable people
- The number of people is a **real** number

If we want to relax these assumptions, and **more closely mimic reality**, we need a more complex model

Example 2: SIR

The **SIR** (Susceptible, Infected, Recovered) model is vastly used to model the **dynamics of an epidemic**

Each infected person will infect a proportion σ of the susceptible people ☹

$$\begin{cases} S(t+1) = S(t) - \sigma \cdot S(t) \cdot I(t) \\ I(t+1) = I(t) + \sigma \cdot S(t) \cdot I(t) - \rho \cdot I(t) \\ R(t+1) = R(t) + \rho \cdot I(t) \end{cases}$$

A proportion ρ of the infected people recovers ☺

- σ: disease spreading rate, $\in [0,1]$
- ρ: recovery rate, $\in [0,1]$

- $S(t)$: number of people that are susceptible to the disease
- $I(t)$: number of infected people
- $R(t)$: number of people that recover for the disease and are not susceptible anymore

Example 2: SIR

$$\Delta S(t+1) = S(t+1) - S(t)$$

$$\begin{cases} S(t+1) = S(t) - \sigma \cdot S(t) \cdot I(t) \\ I(t+1) = I(t) + \sigma \cdot S(t) \cdot I(t) - \rho \cdot I(t) \\ R(t+1) = R(t) + \rho \cdot I(t) \end{cases} \Rightarrow \begin{cases} \Delta S(t+1) = -\sigma \cdot S(t) \cdot I(t) \\ \Delta I(t+1) = \sigma \cdot S(t) \cdot I(t) - \rho \cdot I(t) \\ \Delta R(t+1) = \rho \cdot I(t) \end{cases}$$

$$\Delta I(t+1) = \sigma S(t) I(t) - \rho I(t) \Rightarrow \Delta I(t+1) = I(t)\left[\frac{\sigma}{\rho} S(t) - 1\right]\rho = I(t)[R_0 \cdot S(t) - 1]\rho$$

Basic reproduction ratio $R_0 = \sigma/\rho$

Expected number of new infections from a single infection in a population where all subjects are susceptible

Given $I(t) > 0$:

- $\Delta I(t+1) > 0 \quad \text{IF} \quad R_0 > \frac{1}{S(t)}$
- $\Delta I(t+1) < 0 \quad \text{IF} \quad R_0 < \frac{1}{S(t)}$

$\Delta I(t) = 0$ if:

- $R_0 = \frac{1}{S(t)}$
- $I(t) = 0$

Example 2: Simulation

In order to analyze the phenomena, we can simulate the model

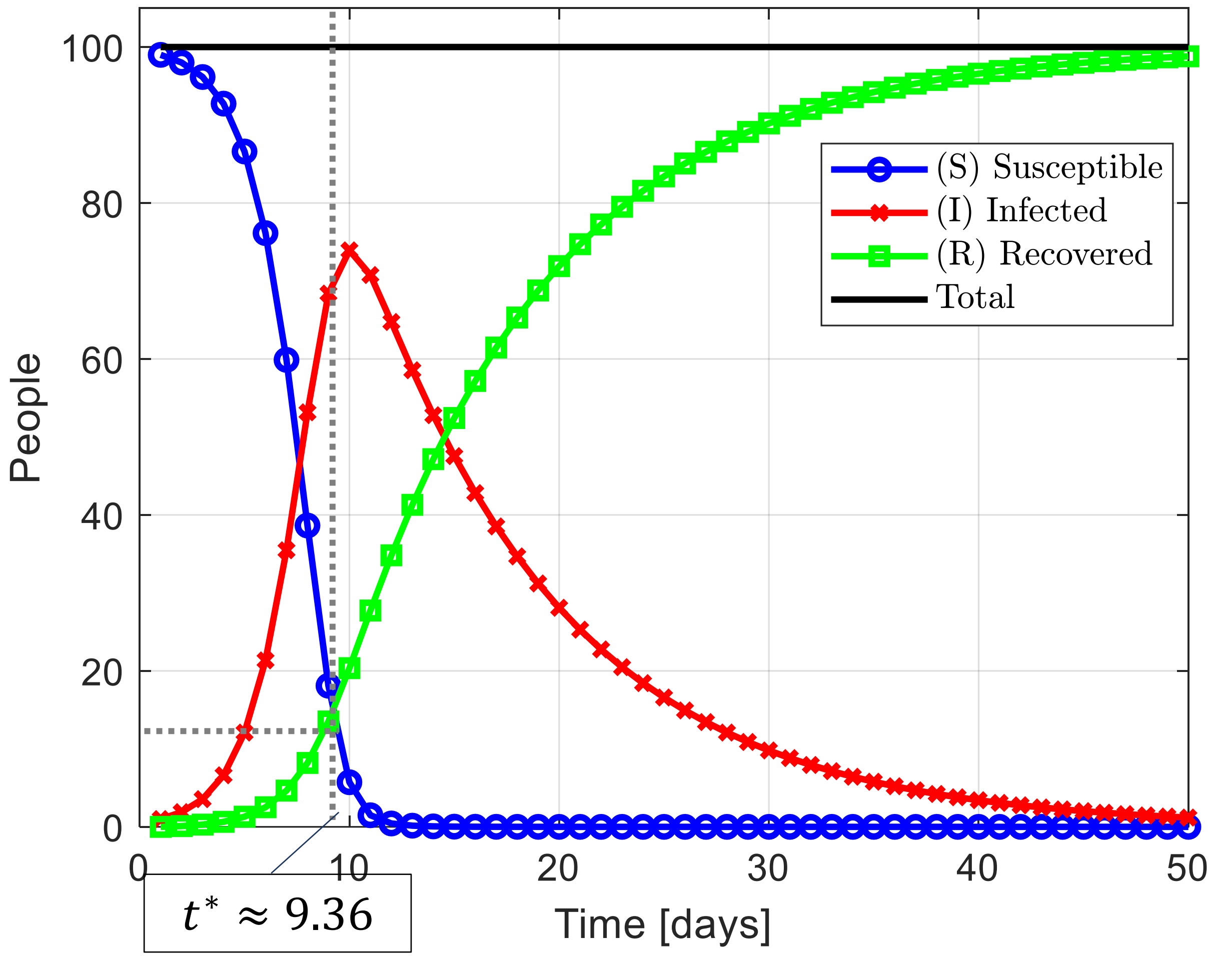
```
sigma = 0.01;
rho = 0.1;
T = 50;

S(1) = 99.9;
I(1) = 0.1;
R(1) = 0;

for t = 2 : 1 : T
    S(t) = S(t-1) - sigma * S(t-1)*I(t-1);
    I(t) = I(t-1) + sigma * S(t-1)*I(t-1) - rho * I(t-1);
    R(t) = R(t-1) + rho * I(t-1);
end
```

Example 1: Simulation

Population size: $100, \sigma = 0.01, \rho = 0.1$



$$R_0 = \frac{\sigma}{\rho} = 0.1 \qquad \frac{1}{S(t^*)} = R_0$$

$\underline{t < t^*}$

$$R_0 > \frac{1}{S(t)} \rightarrow \Delta I(t+1) > 0$$

Increase in the infected

$\underline{t > t^*}$

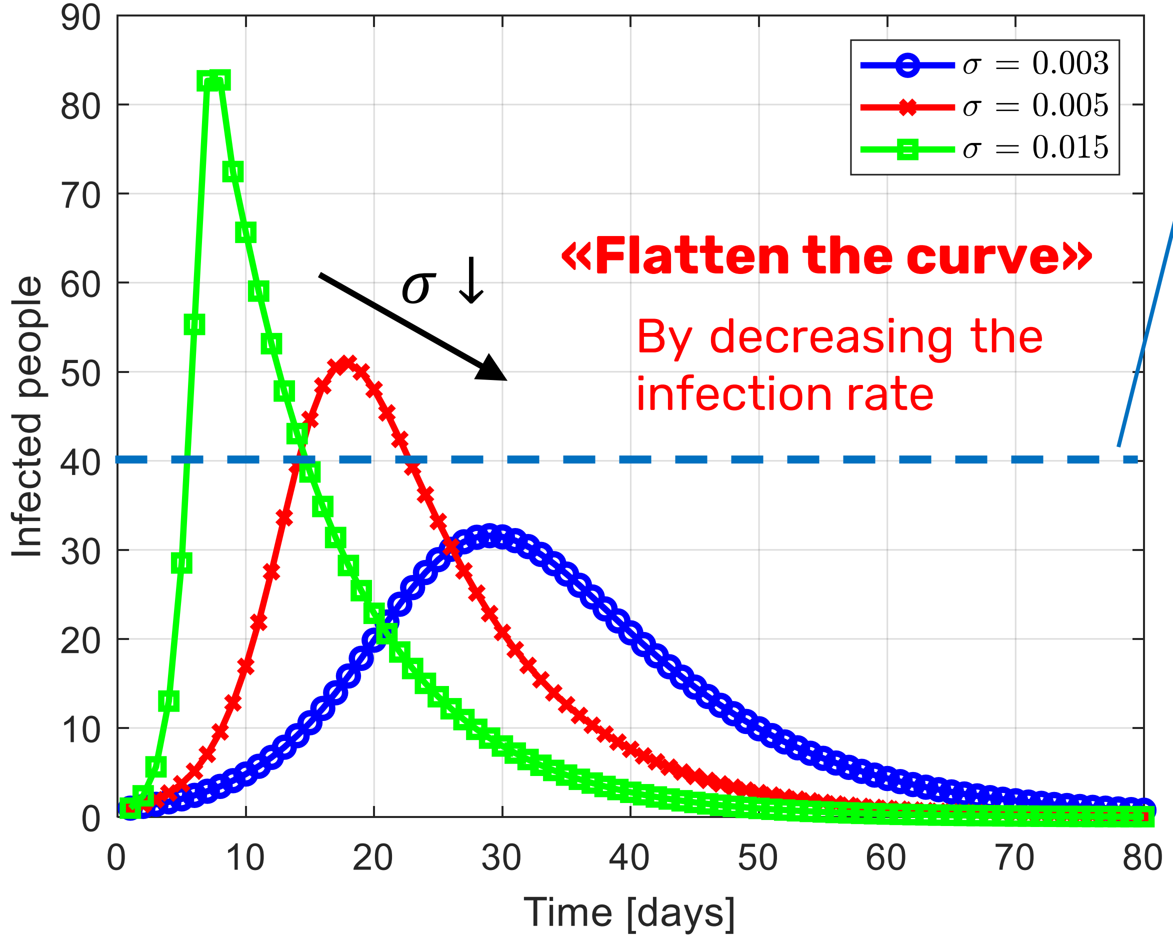
$$R_0 < \frac{1}{S(t)} \rightarrow \Delta I(t+1) < 0$$

Decrease in the infected

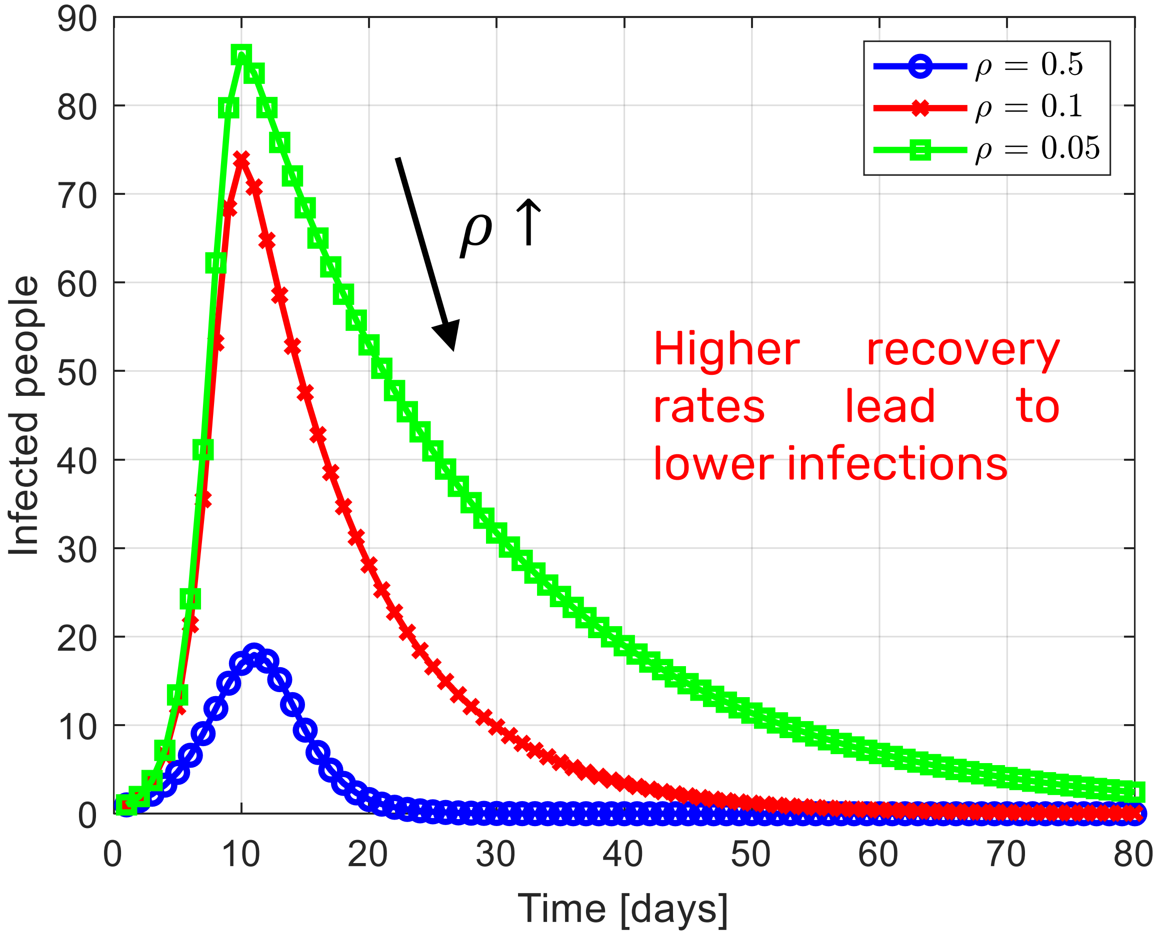
Example 1: Simulation

This could be denote the maximum capacity of the sanitary system

Population size: 100, Different σ, $\rho = 0.1$



Population size: 100, Different ρ, $\sigma = 0.01$



Example 2: Assumptions

The previous model presents the following assumptions:

- The number of persons is a **real number**
- The death/birth rate **is slower** than the infectious disease
- **Everyone can recover (or die)** from the disease

This model can be used to model the seasonal flu outbreak

It is possible to augment the model with **exogenous variables (Inputs)** that vary independently from the model dynamics.

The **inputs will affect** the system behaviour and are usually **known signals**

Example 3: SIR with vaccination

Let's define as $V(t)$ the percentage of **susceptible people that get vaccinated** at the day t.

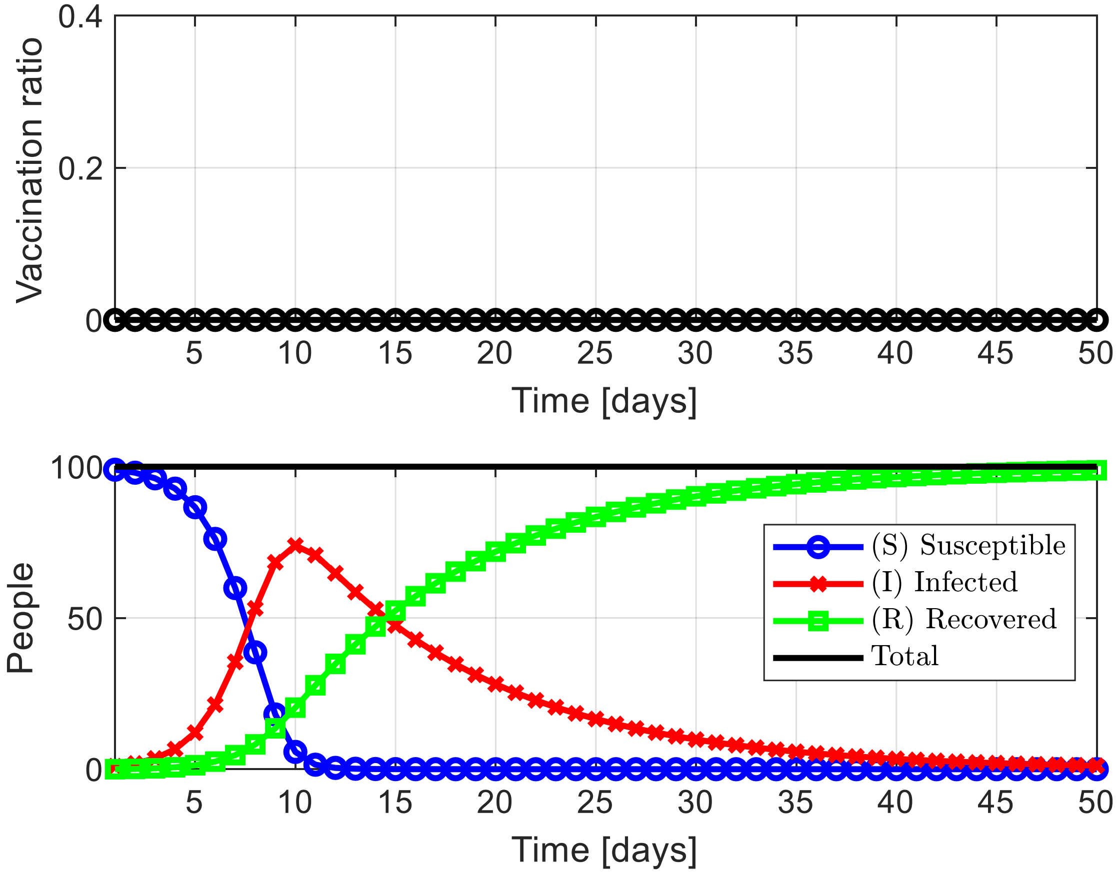
$$\begin{cases} S(t+1) = S(t) - \sigma \cdot S(t) \cdot I(t) - V(t) \cdot S(t) \\ I(t+1) \ = I(t) + \sigma \cdot S(t) \cdot I(t) - \rho \cdot I(t) \\ R(t+1) = R(t) + \rho \cdot I(t) + V(t) \cdot S(t) \end{cases}$$

- σ: disease spreading rate, $\in [0,1]$
- ρ: recovery rate, $\in [0,1]$

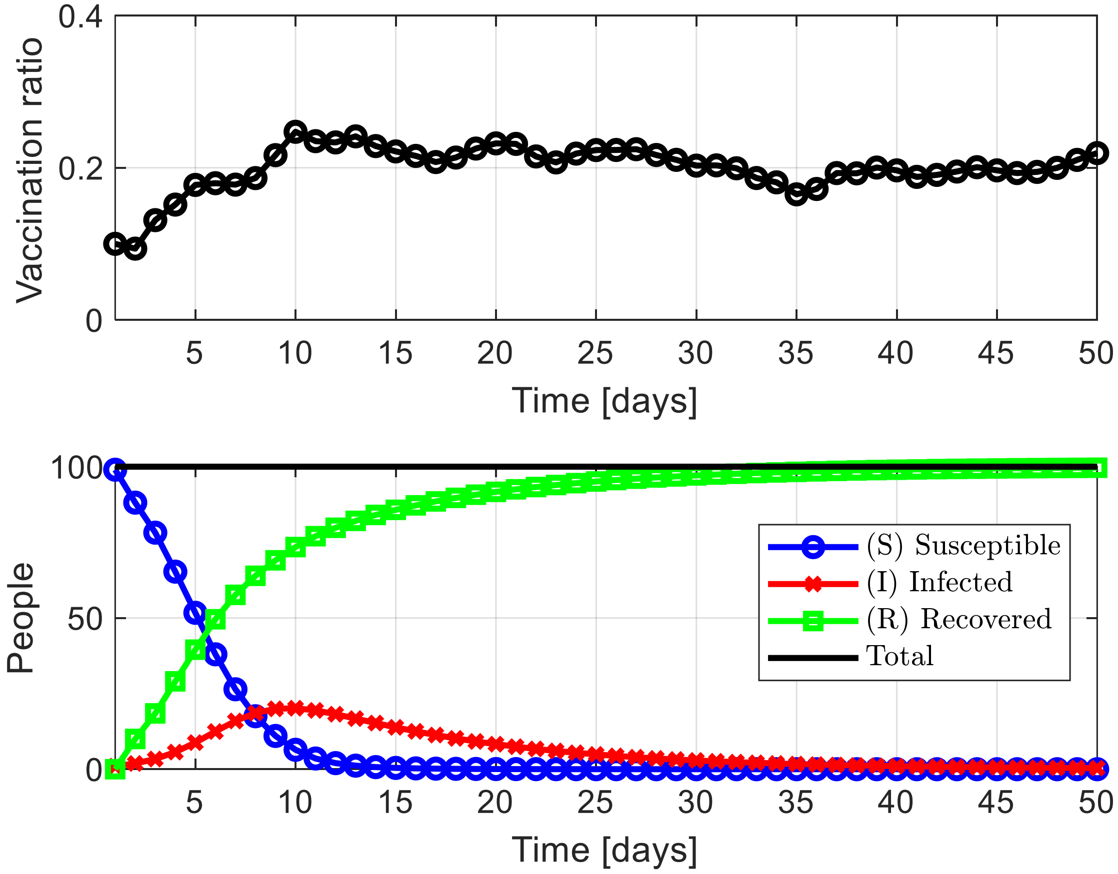
- $S(t)$: number of people that are susceptible to the disease
- $I(t)$: number of infected people
- $R(t)$: number of people that recover for the disease and are not susceptible anymore
- $V(t)$: is an **arbitrary signal** that perturbs the behavior of the model

Example 3: Simulation

Population size: $100, \sigma = 0.01, \rho = 0.1$
No vaccination input



Population size: $100, \sigma = 0.01, \rho = 0.1$
With vaccination input



Latent variables (state variables)

Usually, the system variables are not directly measurable. Thus, we can add an **output equation**, which specifies what we can actually measure.

Suppose that some authority **communicates the number** of infected people $Y(t)$

$$
\begin{cases}
S(t+1) = S(t) - \sigma \cdot S(t) \cdot I(t) - V(t) \cdot S(t) \\
I(t+1) = I(t) + \sigma \cdot S(t) \cdot I(t) - \rho \cdot I(t) \\
R(t+1) = R(t) + \rho \cdot I(t) + V(t) \cdot S(t) \\
Y(t) = \alpha I(t) - \beta\big(S(t) + R(t)\big)
\end{cases}
$$

The output model is a static equation

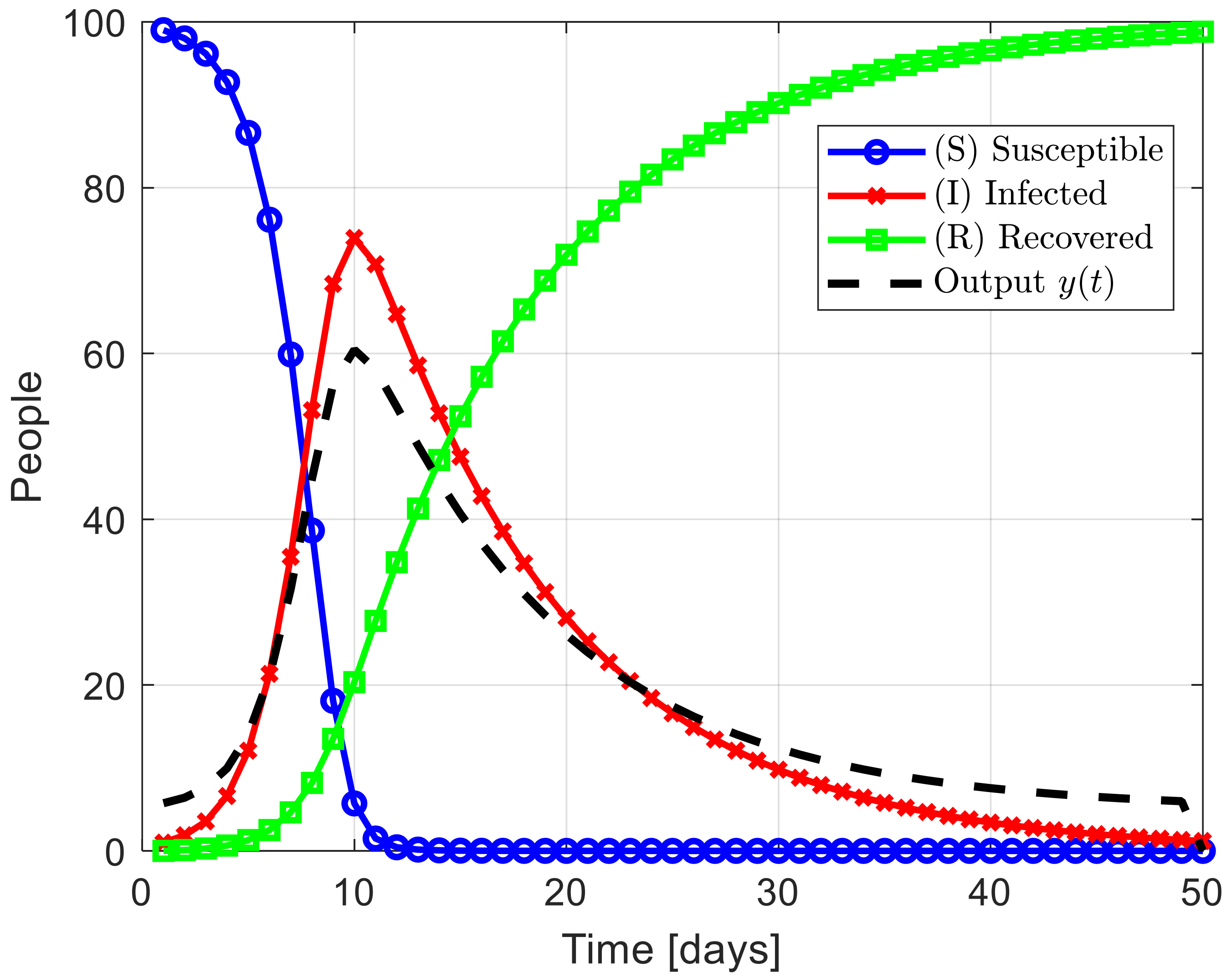
- α: rate of reported cases, $\in [0,1]$
- β: rate of diagnosis errors, $\in [0,1]$

The output at time t can be computed using only information up to time t, and depends on the **state variables** $S(t), I(t), R(t)$ (the model dynamics)

Example 4: Simulation

- **Population size:** $\mathbf{100}$
- $\boldsymbol{\sigma = 0.01, \rho = 0.1, \alpha = 0.8, \beta = 0.05}$
- **No vaccination input**

The output mimic the infected latent state variable $I(t)$, up to a certain **measurement error**



General representation

In general, a discrete-time dynamical system can be written as:

$$\begin{cases} \boldsymbol{x}(t+1) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) \\ \boldsymbol{y}(t) = \boldsymbol{g}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) \end{cases}$$

- $\boldsymbol{x}(t) \in \mathbb{R}^{n \times 1}$ are the **states** or latent variables
- $\boldsymbol{y}(t) \in \mathbb{R}^{p \times 1}$ are the **outputs** or measurements
- $\boldsymbol{u}(t) \in \mathbb{R}^{m_u \times 1}$ are the **inputs** or exogenous variables
- $\boldsymbol{f}(\cdot,\cdot) \in \mathbb{R}^{n \times 1}$ is the **process function**
- $\boldsymbol{g}(\cdot,\cdot) \in \mathbb{R}^{p \times 1}$ is the **output function**

SIR dynamical system example

- $\boldsymbol{x}(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}$
- $\boldsymbol{u}(t) = V(t)$
- $\boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) = \begin{bmatrix} S(t) - \sigma \cdot S(t) \cdot I(t) - V(t) \cdot S(t) \\ I(t) + \sigma \cdot S(t) \cdot I(t) - \rho \cdot I(t) \\ R(t) + \rho \cdot I(t) + V(t) \cdot S(t) \end{bmatrix}$
- $\boldsymbol{y}(t) = Y(t)$
- $\boldsymbol{g}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) = \alpha \cdot I(t) + \beta \cdot \big(S(t) + R(t)\big)$

Linear systems: general representation

When f and g are **linear functions** of the states $\boldsymbol{x}(t)$ and the inputs $\boldsymbol{u}(t)$, we talk of **Linear dynamical systems** *(we consider the discrete-time case)*

$$\begin{cases} \boldsymbol{x}(t+1) = A \cdot \boldsymbol{x}(t) + B \cdot \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \boldsymbol{u}(t) \end{cases}$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \in \mathbb{R}^{n \times n} \quad B = \begin{bmatrix} b_{1,1} & & b_{1,m_u} \\ \vdots & & \\ b_{n,1} & & b_{n,m_u} \end{bmatrix} \in \mathbb{R}^{n \times m_u}$$

$$C = \begin{bmatrix} c_{1,1} & \cdots & c_n \\ & & \\ c_{p,1} & & c_{p,p} \end{bmatrix} \in \mathbb{R}^{p \times n} \quad D = \begin{bmatrix} d_{1,1} & \cdots & d_{1,m_u} \\ & & \\ d_{p,1} & & d_{p,m_u} \end{bmatrix} \in \mathbb{R}^{p \times m_u}$$

Example (SISO system $p = 1, m_u = 1$)

$$\begin{cases} x_1(t+1) = 0.5 \cdot x_1(t) + x_2(t) + 3 \cdot u(t) \\ x_2(t+1) = 0.1 \cdot x_2(t) \\ y(t) = x_1(t) + 3 \cdot x_2(t) + 5 \cdot u(t) \end{cases}$$

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.1 \end{bmatrix} \in \mathbb{R}^{2 \times 2} \quad B = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \in \mathbb{R}^{2 \times 1}$$

$$C = \begin{bmatrix} 1 & 3 \end{bmatrix} \in \mathbb{R}^{1 \times 2} \quad D = 5 \in \mathbb{R}$$

Z-transform

Given a discrete signal $s(t)\colon \mathbb{Z}^+ \to \mathbb{R}$ the Z –transform is the polynomial:

$$\boxed{\mathcal{Z}[s](z) = S(z) \equiv \sum_{t=0}^{\infty} s(t) \cdot z^{-t} = s(0) + s(1)z^{-1} + \cdots}$$

- $z \in \mathbb{C}$ is a complex number

If the series converges (for inputs s.t. $s(t) = 0, \forall t < 0$, this is always true), then $S(z)$ can be espressed as

$$S(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are finite degree polynomials

Delay

Given a signal $s(t)$, then the signal $w(t) = s(t-1)$ has Z-transform:

$$\mathcal{Z}[w](z) = W(z) = \sum_{t=0}^{\infty} w(t) \cdot z^{-t} = w(0) + w(1)z^{-1} + w(2)z^{-2} + \cdots$$

$$= s(-1) + s(0)z^{-1} + s(1)z^{-2} + \cdots$$

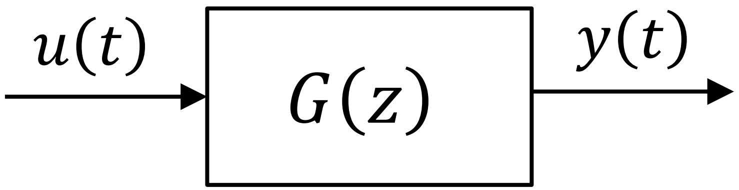
$$= 0 + z^{-1} \cdot (s(0) + s(1)z^{-1} + \cdots)$$

$$= z^{-1} \cdot \sum_{t=0}^{\infty} s(t) \cdot z^{-t} \boxed{= \; z^{-1} \cdot S(z)}$$

We can interpret z^{-1} has a **delay operator** (and z as a **forward operator**)

Linear systems: transfer function

Consider a **SISO** system. The **transfer function** $G(z)$ describes the relation between the **input and output** of a LTI (linear time invariant) dynamical system, when $\boldsymbol{x}(0) = 0$



In a SISO system, it is possible to express $G(z)$ as the **ratio** between the Z-transform of the input signal and the Z-transform of the output signal

$$G(z) = \frac{\mathcal{Z}[y](z)}{\mathcal{Z}[u](z)} = \frac{Y(z)}{U(z)}$$

$\Rightarrow$ Thus, $G(z)$ will be the ratio of **two rational polynomials**

Linear systems: transfer function

In general, if we apply the Z-transform to the state variables, to the inputs and to the outputs, we obtain that $\boldsymbol{G}(z)$ can be expressed as:

$$\boxed{\boldsymbol{G}(z) = C(zI_n - A)^{-1}B + D}$$

(discrete-time case)

$$\boldsymbol{u}(t) \longrightarrow \boxed{\boldsymbol{G}(z)} \longrightarrow \boldsymbol{y}(t)$$

The transfer function can be viewed as an object that **filters** the input $\boldsymbol{u}(t)$ to obtain the output $\boldsymbol{y}(t)$

The filtering behaviour of $\boldsymbol{G}(z)$ can be viewed in the **frequency domain**

The transfer function $\boldsymbol{G}(z)$ depends **only on the system** and not on the **input signal**

Linear systems: transfer function

$$A = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \end{bmatrix} \qquad D = 0$$

$$G(z) = C(zI_n - A)^{-1}B + D$$

$$= \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 =$$

$$= \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} z - 0.1 & -0.4 \\ -0.3 & z - 0.2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a \cdot d - b \cdot c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$= \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \left(\frac{1}{z^2 - 0.3z - 0.1} \begin{bmatrix} z - 0.2 & 0.4 \\ 0.3 & z - 0.1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

Linear systems: transfer function

$$A = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \end{bmatrix} \qquad D = 0$$

$$G(z) = \frac{1}{z^2 - 0.3z - 0.1} \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} z - 0.2 & 0.4 \\ 0.3 & z - 0.1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{z^2 - 0.3z - 0.1} \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} z - 0.2 \\ 0.3 \end{bmatrix} \Rightarrow$$

$$\boxed{G(z) = \frac{3z - 0.3}{z^2 - 0.3z - 0.1}}$$

The denominator is the **characteristic polynomial** of the matrix A

Linear systems: transfer function

Transfer function form

$$\boxed{G(z) = \frac{3z - 0.3}{z^2 - 0.3z - 0.1}}$$

With *a little abuse of notation*, we can write:

$$y(t) = G(z)u(t) = \frac{3z - 0.3}{z^2 - 0.3z - 0.1}u(t) \Rightarrow y(t) = \frac{3z^{-1} - 0.3z^{-2}}{1 - 0.3z^{-1} - 0.1z^{-2}}u(t)$$

$$\boxed{y(t) = 0.3y(t-1) + 0.1y(t-2) + 3u(t-1) - 0.3u(t-2)}$$

Recursive equation (filter) form

Linear systems representation summary

Summarizing, we can represent a **LTI SISO** discrete-time dynamica systems as:

1) State-space representation

$$\begin{cases} x_1(t+1) = 0.1x_1(t) + 0.4x_2(t) + u(t) \\ x_2(t+1) = 0.3x_2(t) + 0.2x_2(t) \\ y(t) = 3x_1(t) + x_2(t) \end{cases}$$

$C(zI_n - A)^{-1}B + D$

realization

2) Transfer function representation

$$G(z) = \frac{3z^{-1} - 0.3z^{-2}}{1 - 0.3z^{-1} - 0.1z^{-2}}$$

3) Recursive filter representation

$$y(t) = 0.3y(t-1) + 0.1y(t-2) + 3u(t-1) - 0.3u(t-2)$$

The state-space is the most complete representation. The transfer function form represents only the states that are reachable\observable from input\output signals, respectively

Linear systems: zeros and poles

The **transfer function polynomials** describe the properties of the dynamical system

- **Zeros:** roots of the numerator
- **Poles:** roots of the denominator (eigenvalues of the A **matrix)**

$$G(z) = \frac{3z - 0.3}{z^2 - 0.3z - 0.1}$$

A discrete-time LTI dynamical system is said to be **asymptotically stable** iff its **poles are in modulus less than 1**

$z^2 - 0.3z - 0.1 \rightarrow \text{Poles: } \; z_1 = 0.5; \;\; z_2 = -0.2$ $\qquad |z_1| < 1 \;\&\&\; |z_2| < 1 \rightarrow$ **asymptotically stable system**

- Asymptotic stability implies that the output of the system has a «bounded energy», given a «bounded energy» input
- If a system is in a stable equilibrium state, it will return to it after a perturbation

Frequency response

Consider a sampled sine wave with the sampling period T_s. The sampled values are:

$$s(t) = A \cdot \sin(2\pi f_0 \cdot t \cdot T_s + \varphi)$$

Amplitude Frequency Phase

With sampling period T_s, the **Nyquist frequency** is: $f_{Nyq} = \frac{f_s}{2} = \frac{1}{2 \cdot T_s}$

In order to sample correctly, we need to use a sufficiently high sampling frequency f_s

The sine frequency has to respect the **Nyquist criteria (sampling theorem)**

$$f_0 \leq f_{Nyq} = \frac{f_s}{2}$$

Frequency response theorem

Consider an asymptotically stable LTI SISO system with transfer function $G(z)$.

$$u(t) \rightarrow \boxed{G(z)} \rightarrow y(t)$$

Consider an input singal $u(t)$ such that: $u(t) = A \cdot \sin(2\pi T_s t \cdot f + \varphi)$

The output signal is: $y(t) = \tilde{y}(t) + \bar{A} \cdot \sin(2\pi T_s t \cdot f + \bar{\varphi})$

Transient effect

$$\boxed{\lim_{t \to \infty} \tilde{y}(t) = 0}$$

System Gain effect

$$\boxed{\bar{A} = A \cdot \left| G\left(e^{j \cdot 2\pi f T_s}\right) \right|}$$

System phase shift

$$\boxed{\bar{\varphi} = \varphi + \angle G\left(e^{j \cdot 2\pi f T_s}\right)}$$

Frequency response theorem

Defining the FRF (Frequency Response Function) of the SISO LTI system as:

$$\boxed{H_{T_S}(f) = G\left(e^{j\cdot 2\pi T_S\cdot f}\right)}$$

We can write the output $y(t)$ given a sine input $u(t) = A \cdot \sin(2\pi T_s t \cdot f + \varphi)$ as :

$$y(t) = A \cdot \left|H_{T_S}(f)\right| \cdot \sin\left(2\pi f \cdot t + \varphi + \angle H_{T_S}(f)\right)$$

The output of an LTI system to a sine wave input, after the transient, is another **sine wave with the same frequency**, but with phase and gain modified by the system

The FRF depends only on the system and the sampling period/frequency

Bode diagrams

The bode diagrams are composed by two graphs:

- The **magnitude graph**, that plots the **magnitude** of the system frequency response

$$\left|H_{T_s}(f)\right|$$

- The **phase graph**, that plots the **phase** of the system frequency response

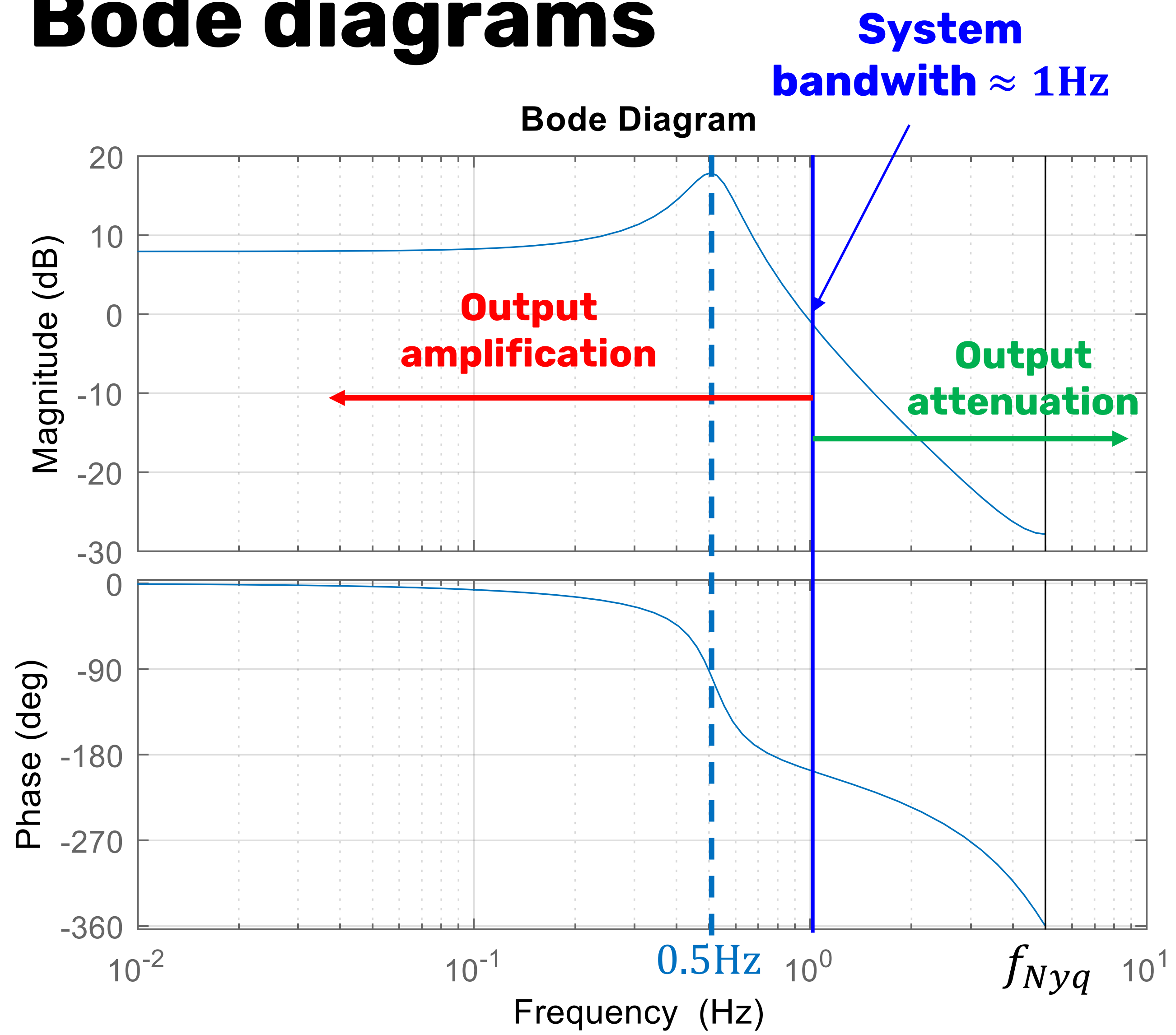
$$\angle H_{T_s}(f)$$

The **frequencies** are plotted in a **logarithmic scale**

The **magnitude** is expressed in **dB (decibels),** i.e. $\mathrm{dB}(f) = 20 \cdot \log\left|H_{T_s}(f)\right|$

The **phase** is usually expressed in **degrees**, but sometimes the radiants are used instead

Bode diagrams

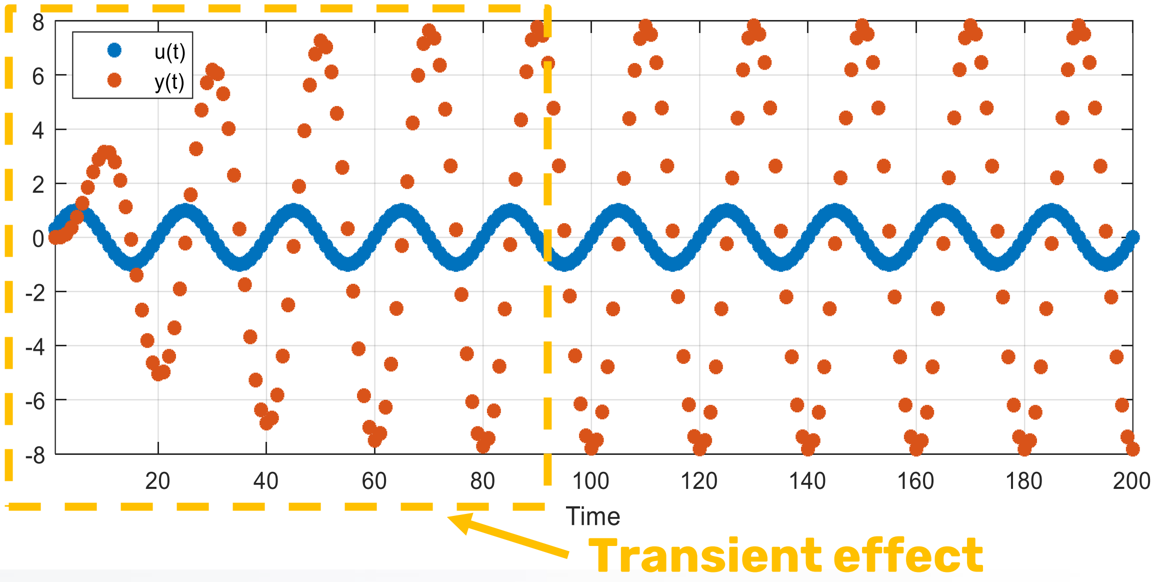


$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

$$T_s = 0.1s \Rightarrow f_{Nyq} = \frac{1}{2T_s} = \frac{1}{2 \cdot 0.1} = 5$$

Compute the output of the system with input (sine wave with frequency 0.5Hz):

$$u(t) = \sin(2\pi T_s t \cdot 0.5)$$



Linear systems: continuous time

Physical systems are naturally described in **continuous time**, i.e. $t \in \mathbb{R}_{>0}$, as opposite to discrete systems where $t \in \mathbb{N}_{>0}$. In this case, a linear dynamical model reads as:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A \cdot \boldsymbol{x}(t) + B \cdot \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \boldsymbol{u}(t) \end{cases} \xrightarrow[\mathcal{L}[\ \ ](s)]{\textbf{Laplace transform}} \begin{cases} s\boldsymbol{X}(s) = A \cdot \boldsymbol{X}(s) + B \cdot \boldsymbol{U}(s) \\ \boldsymbol{Y}(s) = C \cdot \boldsymbol{X}(s) + D \cdot \boldsymbol{U}(s) \end{cases}$$

The transfer function $\boldsymbol{G}(s)$ can be computer by resorting to the **Laplace transformation**, where $s \in \mathbb{C}$ is the Laplace variable. We then have that

(continuous-time case)

$$\boxed{\boldsymbol{G}(s) = C(sI_n - A)^{-1}B + D}$$

- The system $\boldsymbol{G}(s)$ is asymptotically stable iif the poles are < 0

How to get models of dynamical systems

There are several ways to define a model for a physical system:

1. **White-box models:** derive a continuous-time model from the physics of the system, by combining differential equation (usually conservation laws). Then, **discretize** the model with the sampling frequency f_s of your measured signals (Matlab `c2d`)
2. **Gray-box models:** derive the model structure (number of poles\zeros) from physical laws, but estimate its parameters (ex. The transfer fuction polynomial coefficients) from data
3. **Black-box models:** estimate both the model structure and the model parameters from data

IMAD (6 cfu) 1° year Master degree Computer Engineering
Dynamic systems identification (9 cfu) 1° year EMH

Outline

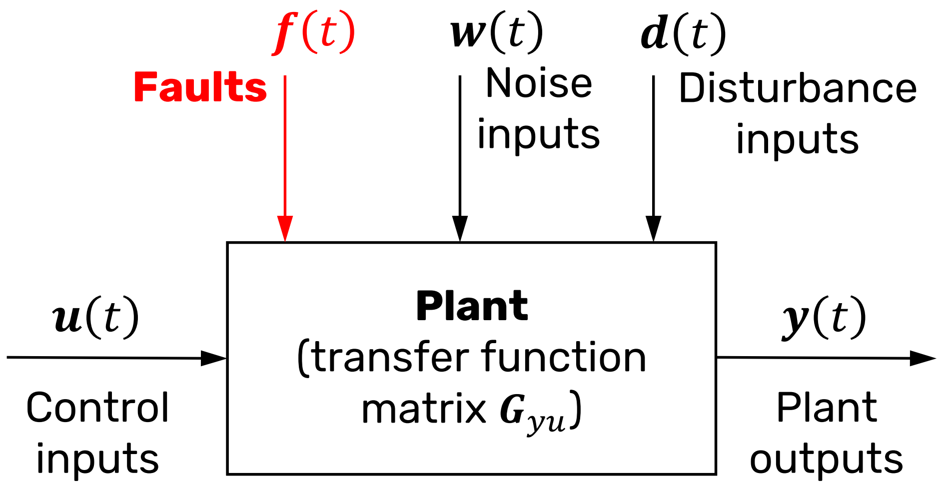
1. Schematic of the approach
2. Dynamical systems
3. **Parity space approach**
4. Diagnostic observer
5. Application to EMA fault detection

Modeling systems with faults

We consider faults that can be modeled as **additive signals**

- Many faults (on **actuators** and **sensors**) can be modeled in this way
- Process faults, usually modeled as multiplicative faults, can be **restated as additive**

$$\begin{cases} \underset{n\times 1}{\boldsymbol{x}(t+1)} = \underset{n\times n}{A} \cdot \boldsymbol{x}(t) + \underset{n\times m_u}{B} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{n\times m_d}{B_d} \cdot \underset{m_d\times 1}{\boldsymbol{d}(t)} + \underset{n\times m_f}{B_f} \cdot \underset{m_f\times 1}{\boldsymbol{f}(t)} \\ \underset{p\times 1}{\boldsymbol{y}(t)} = \underset{p\times n}{C} \cdot \boldsymbol{x}(t) + \underset{p\times m_u}{D} \cdot \boldsymbol{u}(t) + \underset{p\times m_d}{D_d} \cdot \boldsymbol{d}(t) + \underset{p\times m_f}{D_f} \cdot \boldsymbol{f}(t) \end{cases}$$



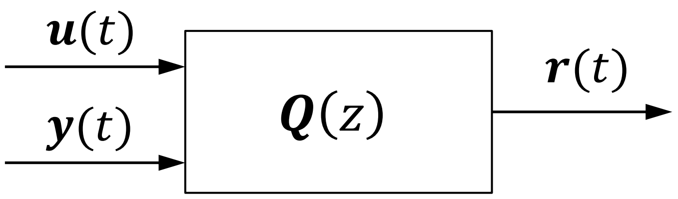
- $\boldsymbol{d}(t) \in \mathbb{R}^{m_d\times 1}$: additive unknown disturbances
- $\boldsymbol{f}(t) \in \mathbb{R}^{m_f\times 1}$: additive fault signals
- We assume no noise signals $\boldsymbol{w}(t)$ are present

The parity space framework

In the parity space FDI (Fault Detection and Isolation) framework the **dynamics** of the **residual signals** are presented in the form of **algebraic equations**

Thus, most the problem solutions are achieved with **linear algebra tools**

- the system designer is not required to have rich knowledge of advanced control theory
- most computations can be completed without complex and involved algorithms
- a great number of FDI methods and ideas have been first presented in the parity space framework, and later extended to other frameworks



Parity space idea

Suppose that $\boldsymbol{w}(t) = \mathbf{0}$, $\boldsymbol{d}(t) = \mathbf{0}$ and $\boldsymbol{f}(t) = \mathbf{0}$, i.e. the system is **not subject** to external noise, disturbances and faults. Then, with $s \geq 0$, the following relations hold:

- $\boldsymbol{y}(t-s) = C\boldsymbol{x}(t-s) + D\boldsymbol{u}(t-s)$

- $\boldsymbol{y}(t-s+1) = C\boldsymbol{x}(t-s+1) + D\boldsymbol{u}(t-s+1)$

$$= C \cdot \{A\boldsymbol{x}(t-s) + B\boldsymbol{u}(t-s)\} + D\boldsymbol{u}(t-s+1)$$

$$= CA^{1}\boldsymbol{x}(t-s) + CB\boldsymbol{u}(t-s) + D\boldsymbol{u}(t-s+1)$$

Parity space idea

- $\boldsymbol{y}(t-s+1) = C\boldsymbol{x}(t-s+1) + D\boldsymbol{u}(t-s+1)$

$$= C \cdot \{A\boldsymbol{x}(t-s) + B\boldsymbol{u}(t-s)\} + D\boldsymbol{u}(t-s+1)$$

$$= CA^{1}\boldsymbol{x}(t-s) + CB\boldsymbol{u}(t-s) + D\boldsymbol{u}(t-s+1)$$

- $\boldsymbol{y}(t-s+2) = C\boldsymbol{x}(t-s+2) + D\boldsymbol{u}(t-s+2)$

$$= C \cdot \{A\boldsymbol{x}(t-s+1) + B\boldsymbol{u}(t-s+1)\} + D\boldsymbol{u}(t-s+2)$$

$$= CA \cdot \{A\boldsymbol{x}(t-s) + B\boldsymbol{u}(t-s)\} + CB\boldsymbol{u}(t-s+1) + D\boldsymbol{u}(t-s+2)$$

$$= CA^{2}\boldsymbol{x}(t-s) + CA^{2-1}B\boldsymbol{u}(t-s) + CB\boldsymbol{u}(t-s+1) + D\boldsymbol{u}(t-s+2)$$

$t-s+(2-1)$ $\quad$ $t-s+(2)$

Parity space idea

- $\boldsymbol{y}(t-s+2) = C\boldsymbol{x}(t-s+2) + D\boldsymbol{u}(t-s+2)$

$$= C \cdot \{A\boldsymbol{x}(t-s+1) + B\boldsymbol{u}(t-s+1)\} + D\boldsymbol{u}(t-s+2)$$

$$= CA \cdot \{A\boldsymbol{x}(t-s) + B\boldsymbol{u}(t-s)\} + CB\boldsymbol{u}(t-s+1) + D\boldsymbol{u}(t-s+2)$$

$$= CA^2\boldsymbol{x}(t-s) + CA^{2-1}B\boldsymbol{u}(t-s) + CB\boldsymbol{u}(\underbrace{t-s+1}_{t-s+(2-1)}) + D\boldsymbol{u}(\underbrace{t-s+2}_{t-s+(2)})$$

$\vdots$

- $\boldsymbol{y}(t-s+s) = \boldsymbol{y}(t) = CA^s\boldsymbol{x}(t-s) + CA^{s-1}B\boldsymbol{u}(t-s) + \cdots + CB\boldsymbol{u}(\underbrace{t-1}_{t-s+(s-1)}) + D\boldsymbol{u}(\underbrace{t}_{t-s+(s)})$

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Parity space idea

Define now the following quantities:

$$\underset{m_u(s+1)\times 1}{\boldsymbol{u}_s(t)} = \begin{bmatrix} \underset{m_u\times 1}{\boldsymbol{u}(t-s)} \\ \boldsymbol{u}(t-s+1) \\ \vdots \\ \boldsymbol{u}(t) \end{bmatrix} \qquad \underset{p(s+1)\times 1}{\boldsymbol{y}_s(t)} = \begin{bmatrix} \underset{p\times 1}{\boldsymbol{y}(t-s)} \\ \boldsymbol{y}(t-s+1) \\ \vdots \\ \boldsymbol{y}(t) \end{bmatrix}$$

$$\underset{p(s+1)\times m_u(s+1)}{H_{u,s}} = \begin{bmatrix} \overset{p\times m_u}{D} & 0 & \cdots & \cdots & 0 \\ CB & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ CA^{s-1}B & \cdots & \cdots & CB & D \end{bmatrix} \qquad \underset{p(s+1)\times n}{H_{o,s}} = \begin{bmatrix} \underset{p\times n}{C} \\ CA \\ \vdots \\ CA^s \end{bmatrix}$$

Parity space idea

Given the quantities $\boldsymbol{u}_s$, $\boldsymbol{y}_s$, $H_{o,s}$, $H_{u,s}$, we can write

$$\boxed{\boldsymbol{y}_s(t) = H_{o,s}\boldsymbol{x}(t-s) + H_{u,s}\boldsymbol{u}_s(t)}$$

Parity relation

$$\underbrace{\begin{bmatrix} \boldsymbol{y}(t-s) \\ \boldsymbol{y}(t-s+1) \\ \vdots \\ \boldsymbol{y}(t) \end{bmatrix}}_{p(s+1)\times 1} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^s \end{bmatrix}}_{p(s+1)\times n} \cdot \underbrace{\boldsymbol{x}(t-s)}_{n\times 1} + \underbrace{\begin{bmatrix} D & 0 & \cdots & \cdots & 0 \\ CB & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ CA^{s-1}B & \cdots & \cdots & CB & D \end{bmatrix}}_{p(s+1)\times m_u(s+1)} \cdot \underbrace{\begin{bmatrix} \boldsymbol{u}(t-s) \\ \boldsymbol{u}(t-s+1) \\ \vdots \\ \boldsymbol{u}(t) \end{bmatrix}}_{m_u(s+1)\times 1}$$

Parity space idea

$$\boxed{\boldsymbol{y}_s(t) = H_{o,s}\boldsymbol{x}(t-s) + H_{u,s}\boldsymbol{u}_s(t)}$$

Parity relation

- Describes the **inputs and outputs relationship** based on the past state vector $\boldsymbol{x}(t-s)$
- $\boldsymbol{y}_s$ and $\boldsymbol{u}_s$ consist of the temporal and past outputs and inputs, and are **known**
- Matrices $H_{o,s}$ and $H_{u,s}$ are composite of system matrices A, B, C, D and are also **known**
- The only **unknown** variable is $\boldsymbol{x}(t-s)$

Parity space idea

The underlying idea of the parity relation based residual generation lies in the utilization of the that, for $s \geq n$, the following **rank condition** holds:

$$\operatorname{rank}\left(H_{o,s}\right) \leq n < \text{the number of rows of the matrix } H_{o,s}, \text{i.e. } p(s+1)$$

$$\underset{p(s+1)\times n}{H_{o,s}} = \begin{bmatrix} \underset{p\times n}{C} \\ CA \\ \vdots \\ CA^s \end{bmatrix}$$

Therefore, for $s \geq n$, there **exists** a vector $\boldsymbol{v}_s^T \in \mathbb{R}^{1\times p(s+1)}$, $\boldsymbol{v}_s \neq \mathbf{0}$, such that:

$$\boxed{\boldsymbol{v}_s^T \cdot H_{o,s} = \mathbf{0}}$$

The multiplication of a row vector for a matrix results into a linear combination of the matrix rows

that is, it is possible to express a row of $H_{o,s}$ as **linear combination** of other rows of $H_{o,s}$.

The vector $\boldsymbol{v}_s$ can be **found by solving** the above **linear system**. However, $\boldsymbol{v}_s$ is not guaranteed to be unique.

Parity space residual generator

From the observation that $\boldsymbol{v}_s^T H_{o,s} = \mathbf{0}$, a residual signal $r(t)$ is built as:

$$\boxed{r(t) = \underset{1 \times p(s+1)}{\boldsymbol{v}_s^T} \cdot \underset{p(s+1) \times 1}{\left(\boldsymbol{y}_s(t) - H_{u,s}\, \boldsymbol{u}_s(t)\right)}}$$

Residual generator

In the nominal case where $\boldsymbol{d}(t) = \mathbf{0}, \boldsymbol{f}(t) = \mathbf{0}$, we have that:

$$r(t) = \boldsymbol{v}_s^T \cdot \left(H_{o,s}\, \boldsymbol{x}(t-s)\right) = 0$$

In the nominal case, the **residual is zero**

Vectors satisfying $\boldsymbol{v}_s^T \cdot H_{o,s} = \mathbf{0}$ are called **parity vectors**. The set $P_s = \{\boldsymbol{v}_s \mid \boldsymbol{v}_s H_{o,s} = \mathbf{0}\}$ is called **parity space**

Parity space residual generator

$$\underset{p(s+1)\times n}{H_{o,s}} = \begin{bmatrix} \underset{p\times n}{C} \\ CA \\ \vdots \\ CA^s \end{bmatrix}$$

The main observation for building a parity vector was the fact that:

$$\mathrm{rank}\left(H_{o,s}\right) \leq n < \text{the number of rows of the matrix } H_{o,s}, \text{i. e. } p(s+1)$$

The previous statement can be proved by considering the square matrix $A \in \mathbb{R}^{n\times n}$, its associated **characteristic polynomial** (with $\lambda \in \mathbb{C}$)

$$\varphi(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

and **characteristic equation** $\varphi(\lambda) = 0$, whose solutions are the **eigenvalues** of A

From the **Cayley-Hamilton theorem**, it holds that:

$$\varphi(A) = A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI_n = 0$$

Parity space residual generator

$$\varphi(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I_n = 0$$

$$C \cdot \varphi(A) = CA^n + a_1 CA^{n-1} + a_2 CA^{n-2} + \cdots + a_n C = 0$$

As an example, consider a **2° order LTI SISO** system, i.e. with $A \in \mathbb{R}^{2\times 2}$ and $C \in \mathbb{R}^{1\times 2}$

- $C = [c_1 \quad c_2]$
- $CA = [w_1 \quad w_2]$

$$\underset{3\times 2}{H_{o,s}} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix},$$

$$C \cdot \varphi(A) = CA^2 + a_1 CA + a_2 C = 0$$

$$CA^2 = -a_1 CA - a_2 C = -a_1 [w_1 \quad w_2] - a_2 [c_1 \quad c_2] = [-a_1 \quad -a_2] \begin{bmatrix} w_1 & w_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= [-a_1 \quad -a_2] \begin{bmatrix} CA \\ C \end{bmatrix}$$

Thus, the rows of CA^2 can be expressed as a linear combination of the rows of C and CA, with $\mathrm{rank}(H_{o,s}) \leq 2$

Parity space: example

Consider the SISO nominal system model

$$Y(z) = G_{yu}(z)U(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} U(z)$$

A trivial way to construct a parity space based residual generator is to:

1. rewrite the system into its minimum state space realization form
2. solve the linear system $\boldsymbol{v}_s^T \cdot H_{o,s} = \mathbf{0}$ for $\boldsymbol{v}_s$
3. construct the residual generator $r(t) = \boldsymbol{v}_s^T \cdot \left(\boldsymbol{y}_s(t) - H_{u,s}\, \boldsymbol{u}_s(t)\right)$

Parity space: example

On the other side, it follows from Cayley–Hamilton theorem that:

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_0 I_n = \mathbf{0} \quad \Longrightarrow \quad [a_0 \quad \cdots \quad a_{n-1} \quad 1]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^n \end{bmatrix} = \mathbf{0}$$

where A, C denote the system matrices of the minimum state space realization of $G_{yu}(z)$.

That means that:

$$\boldsymbol{v}_s = [a_0 \quad \cdots \quad a_{n-1} \quad 1]$$

is a parity space vector for $G_{yu}(z)$.

Parity space: example

The residual generator can be constructed as:

$$r(t) = \boldsymbol{v}_s^T \cdot \left(\boldsymbol{y}_s(t) - H_{u,s}\,\boldsymbol{u}_s(t)\right) = \boldsymbol{v}_s^T \boldsymbol{y}_s(t) - \boldsymbol{v}_s^T H_{u,s}\,\boldsymbol{u}_s(t) \ = [a_0 \quad \cdots \quad a_{n-1} \quad 1]\boldsymbol{y}_s(t) - \boldsymbol{v}_s^T H_{u,s}\,\boldsymbol{u}_s(t)$$

$$= a_0 y(t-s) + \cdots + a_1 y(t-s+1) + y(t) - \boldsymbol{v}_s^T H_{u,s}\,\boldsymbol{u}_s(t) = 0$$

For the equality $r(t) = 0$ to hold in the nominal case, it follows that

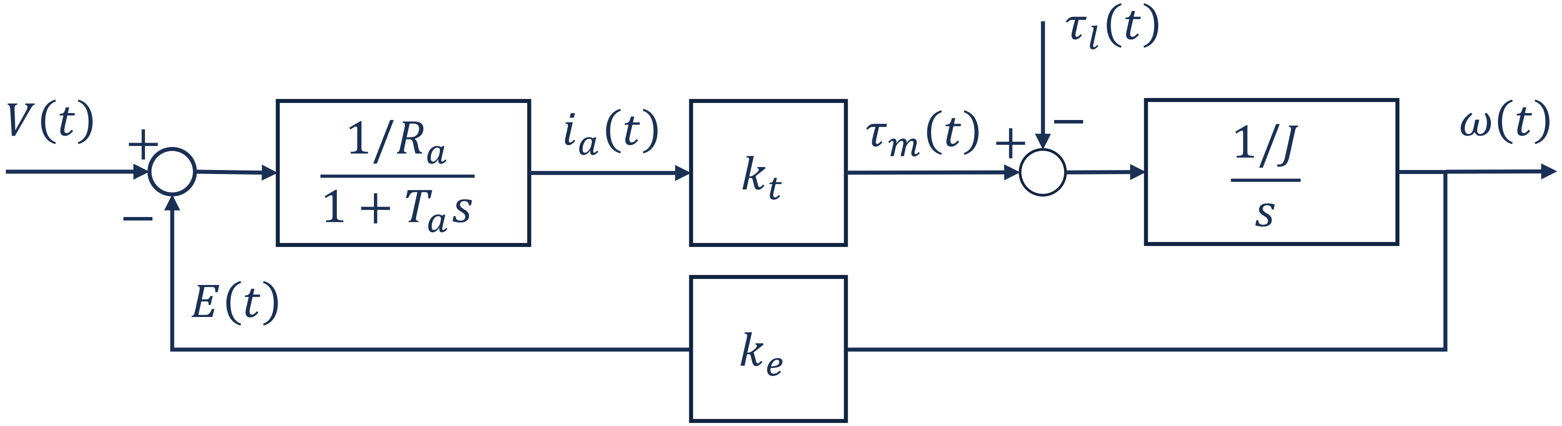
$$\boldsymbol{v}_s^T H_{u,s} = [b_0 \quad \cdots \quad b_{n-1} \quad b_n]$$

As a result, the residual generator, corresponding to the previous choice of $\boldsymbol{v}_s$, is given by

$$\boxed{r(t) = [a_0 \quad \cdots \quad a_{n-1} \quad 1] \cdot \boldsymbol{y}_s\,(t) - [b_0 \quad \cdots \quad b_{n-1} \quad b_n] \cdot \boldsymbol{u}_s\,(t)}$$

Parity space: DC motor example

Consider an open-loop model of a DC motor



- Model input $u(t) = V(t)$
- Model output $y(t) = \omega(t)$
- Disturbance $d(t) = \tau_l(t)$

- Total inertia J: $80.45 \cdot 10^{-6}\ \mathrm{Kg} \cdot \mathrm{m}^2$
- Motor electrical constant k_e: $6.27 \cdot 10^{-3}\ \mathrm{V/rpm}$
- Motor torque constant $k_t = 0.06\ \mathrm{Nm/A}$
- Motor coil inductance $L_a = 0.003\ \mathrm{H}$
- Motor coil resistance $R_a = 3.13\ \Omega$
- Electrical time constant $T_a = L_a/R_a$ s

$$G_{yu}(s) = \frac{1}{K_e \cdot \left(1 + J \cdot \frac{R_a}{K_t K_e} s + J \cdot T_a \cdot \frac{R_a}{K_t K_e} s^2\right)}$$

$$G_{yd}(s) = -\frac{R_a(1 + T_a s)}{K_t K_e \cdot \left(1 + J \cdot \frac{R_a}{K_t K_e} s + J \cdot T_a \cdot \frac{R_a}{K_t K_e} s^2\right)}$$

Parity space: DC motor example

Compute the transfer functions and convert to discrete time (with sampling freq. 100 Hz)

$$G_{yu}(s) = \frac{2.486 \cdot 10^5}{(s + 1042)(s + 1.496)} \xrightarrow{\texttt{c2d}} G_{yu}(z) = \frac{1.184\, z + 1.184}{z^2 - 0.9852\, z + 2.943 \cdot 10^{-5}}$$

$$G_{yd}(s) = \frac{-12430 \cdot (s + 1043)}{(s + 1042)(s + 1.496)} \xrightarrow{\texttt{c2d}} G_{yd}(z) = \frac{-123.6z + 0.003637}{z^2 - 0.9852\, z + 2.943 \cdot 10^{-5}}$$

A **valid parity vector** is:

$$\boldsymbol{v}_s^T = [2.943 \cdot 10^{-5} \quad -0.9852 \quad 1]$$

Parity space: DC motor example

A state space realization of $G_{yu}(z)$ leads to

$$G_{yu}(z) = \frac{1.184\,z + 1.184}{z^2 - 0.9852\,z + 2.943 \cdot 10^{-5}} \xrightarrow{\texttt{minreal(ss(G\_yu))}}$$

$$\underset{\substack{n \times n \\ 2 \times 2}}{A} = \begin{bmatrix} 0.9852 & -0.007535 \\ 0.003906 & 0 \end{bmatrix} \qquad \underset{\substack{n \times m_u \\ 2 \times 1}}{B} = \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

$$\underset{\substack{p \times n \\ 1 \times 2}}{C} = \begin{bmatrix} 0.07401 & 18.95 \end{bmatrix} \qquad \underset{\substack{p \times m_u \\ 1 \times 1}}{D} = 0$$

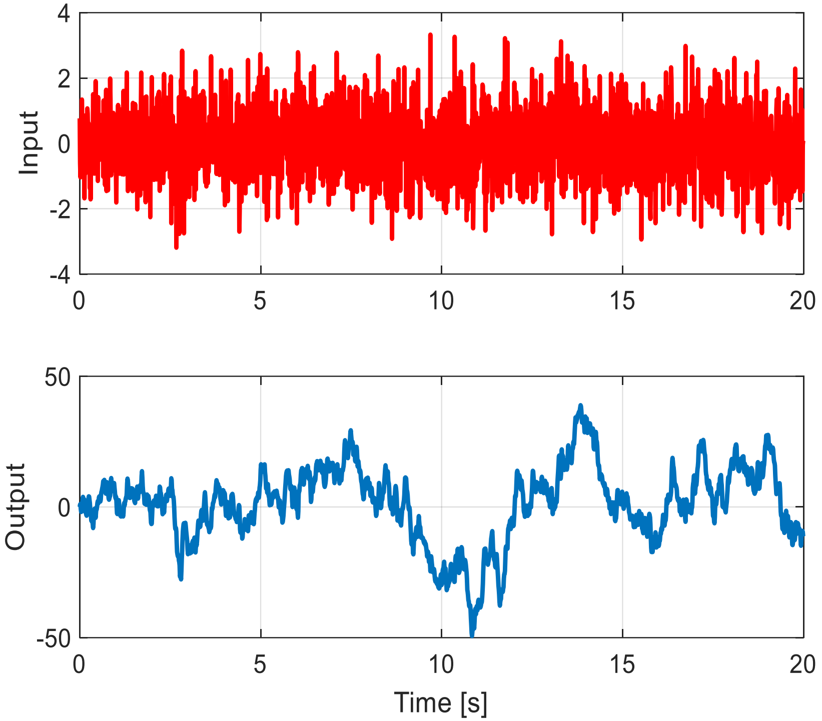
$$\underset{\substack{p(s+1) \times n \\ 1(2+1) \times 2}}{H_{o,s}} = \begin{bmatrix} 0.0740 & 18.9468 \\ 0.1469 & -0.0006 \\ 0.1447 & -0.0011 \end{bmatrix} \qquad \underset{\substack{p(s+1) \times m_u(s+1) \\ 1(2+1) \times 1(2+1)}}{H_{u,s}} = \begin{bmatrix} 0 & 0 & 0 \\ 1.1842 & 0 & 0 \\ 2.3508 & 1.1842 & 0 \end{bmatrix}$$

Always check the solution!

$$\boldsymbol{v}_s^T H_{o,s} = \begin{bmatrix} 2.943 \cdot 10^{-5} & -0.9852 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.0740 & 18.9468 \\ 0.1469 & -0.0006 \\ 0.1447 & -0.0011 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$ **CORRECT!** ☺

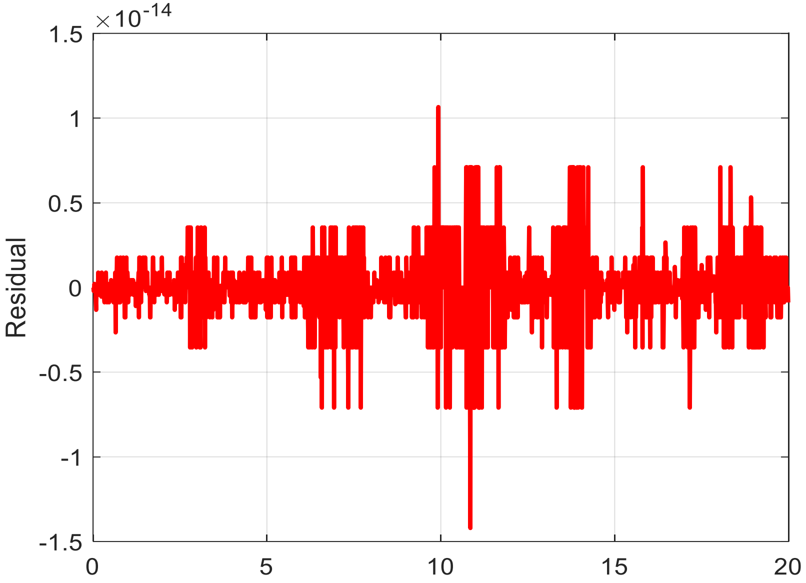
Parity space: DC motor example

1. Simulate data, $u(t) = WN(0,1)$



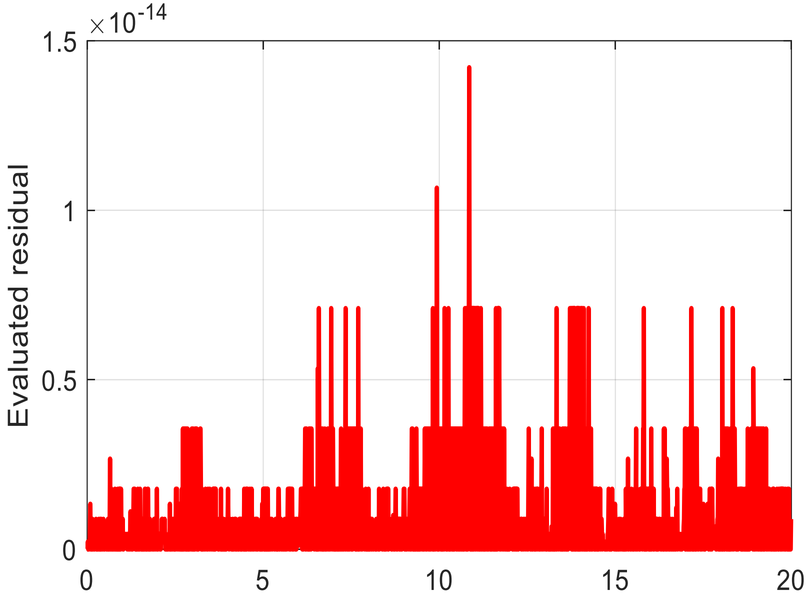
2. Compute the residual

$$r(t) = \boldsymbol{v}_s^T \cdot \left(\boldsymbol{y}_s(t) - H_{u,s}\,\boldsymbol{u}_s(t)\right)$$



3. Evaluate the residual

$$\theta(t) = |r(t)|$$



Parity space: effects of faults and disturbances

Suppose now that disturbances and faults affect the system, i.e. $\boldsymbol{d}(t) \neq 0$ and $\boldsymbol{f}(t) \neq 0$.

By defining the quantities:

$$\underset{m_f(s+1)\times 1}{\boldsymbol{f}_s(t)} = \begin{bmatrix} \underset{m_f \times 1}{\boldsymbol{f}(t-s)} \\ \boldsymbol{f}(t-s+1) \\ \vdots \\ \boldsymbol{f}(t) \end{bmatrix} \qquad \underset{p(s+1)\times m_f(s+1)}{H_{f,s}} = \begin{bmatrix} D_f & 0 & \cdots & \cdots & 0 \\ \underset{p \times m_f}{CB_f} & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ CA^{s-1}B_f & \cdots & \cdots & CB_f & D_f \end{bmatrix}$$

$$\underset{m_d(s+1)\times 1}{\boldsymbol{d}_s(t)} = \begin{bmatrix} \underset{m_d \times 1}{\boldsymbol{d}(t-s)} \\ \boldsymbol{d}(t-s+1) \\ \vdots \\ \boldsymbol{d}(t) \end{bmatrix} \qquad \underset{p(s+1)\times m_d(s+1)}{H_{d,s}} = \begin{bmatrix} D_d & 0 & \cdots & \cdots & 0 \\ \underset{p \times m_d}{CB_d} & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ CA^{s-1}B_d & \cdots & \cdots & CB_d & D_d \end{bmatrix}$$

Parity space: effects of faults and disturbances

Given the quantities $\boldsymbol{u}_s$, $\boldsymbol{y}_s$, $\boldsymbol{f}_s$, $\boldsymbol{d}_s$, $H_{o,s}$, $H_{u,s}$, $H_{f,s}$, $H_{d,s}$ we can write

$$\boldsymbol{y}_s(t) = H_{o,s}\boldsymbol{x}(t-s) + H_{u,s}\boldsymbol{u}_s(t) + H_{d,s}\boldsymbol{d}_s(t) + H_{f,s}\boldsymbol{f}_s(t)$$

$$\begin{bmatrix} \boldsymbol{y}(t-s) \\ \boldsymbol{y}(t-s+1) \\ \vdots \\ \boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^s \end{bmatrix} \cdot \boldsymbol{x}(t-s) + \begin{bmatrix} D & 0 & \cdots & \cdots & 0 \\ CB & & & & \vdots \\ & & & & \vdots \\ \vdots & & & & 0 \\ CA^{s-1}B & \cdots & \cdots & CB & D \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}(t-s) \\ \boldsymbol{u}(t-s+1) \\ \vdots \\ \boldsymbol{u}(t) \end{bmatrix} +$$

$$+ \begin{bmatrix} D_d & 0 & \cdots & \cdots & 0 \\ CB_d & & & & \vdots \\ & & & & \vdots \\ \vdots & & & & 0 \\ CA^{s-1}B_d & \cdots & \cdots & CB_d & D_d \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{d}(t-s) \\ \boldsymbol{d}(t-s+1) \\ \vdots \\ \boldsymbol{d}(t) \end{bmatrix} + \begin{bmatrix} D_f & 0 & \cdots & \cdots & 0 \\ CB_f & & & & \vdots \\ & & & & \vdots \\ \vdots & & & & 0 \\ CA^{s-1}B_f & \cdots & \cdots & CB_f & D_f \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{f}(t-s) \\ \boldsymbol{f}(t-s+1) \\ \vdots \\ \boldsymbol{f}(t) \end{bmatrix}$$

Parity space: effects of faults and disturbances

The effects of the disturbances and faults on the residual are then given by:

$$r(t) = \boldsymbol{v}_s^T \cdot \left(\boldsymbol{y}_s(t) - H_{u,s}\, \boldsymbol{u}_s(t) \right)$$

$$= \boldsymbol{v}_s^T \cdot \left(H_{o,s} \boldsymbol{x}(t-s) + H_{d,s} \boldsymbol{d}_s(t) + H_{f,s} \boldsymbol{f}_s(t) \right)$$

$$\boxed{r(t) = \boldsymbol{v}_s^T \cdot \left(H_{d,s} \boldsymbol{d}_s(t) + H_{f,s} \boldsymbol{f}_s(t) \right)}$$

Parity space: effects of faults and disturbances

$$\boxed{r(t) = \boldsymbol{v}_s^T \cdot \left(H_{d,s}\boldsymbol{d}_s(t) + H_{f,s}\boldsymbol{f}_s(t)\right)}$$

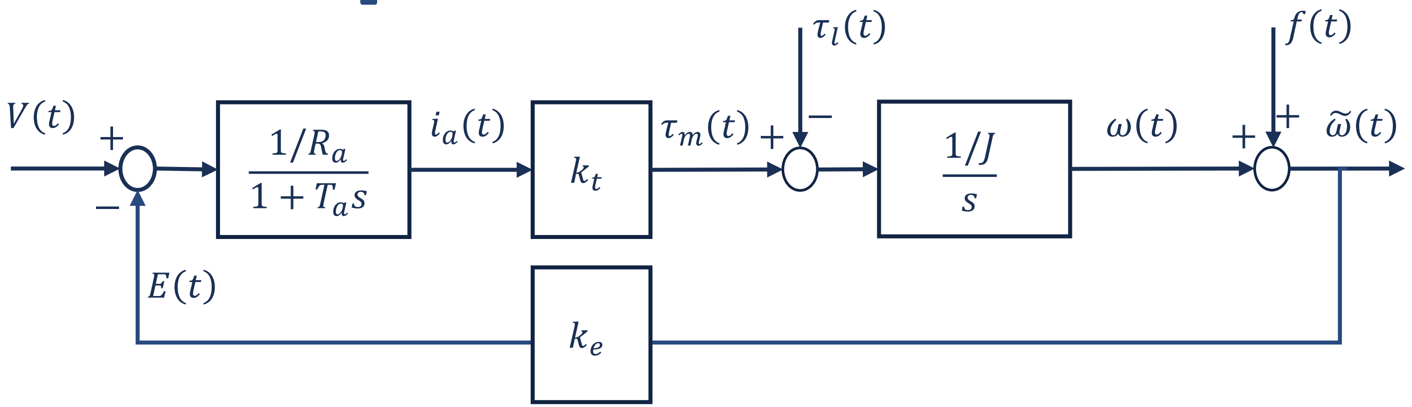
Ideally, if $\boldsymbol{v}_s^T H_{f,s} \neq \mathbf{0}$, the residual is **not zero** when a fault is present, and so **fault detection is achieved**. However, the residual is also sensitive to the disturbances.

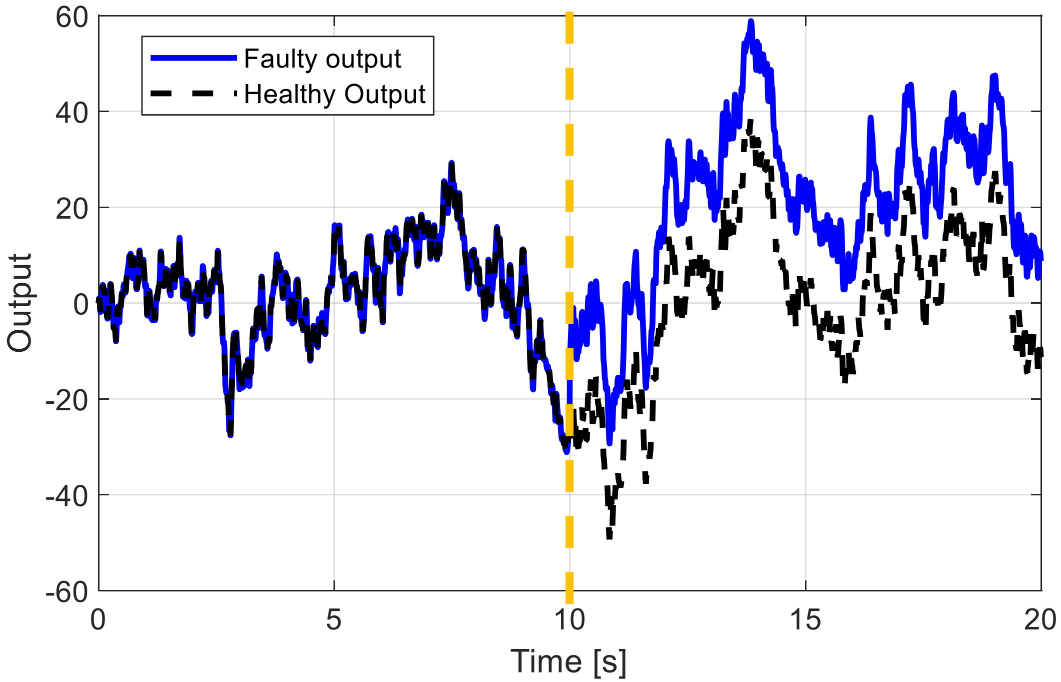
The **choice of the parity vector** has decisive impact on the performance of the residual generator. Its design can however be carried out in a straightforward manner.

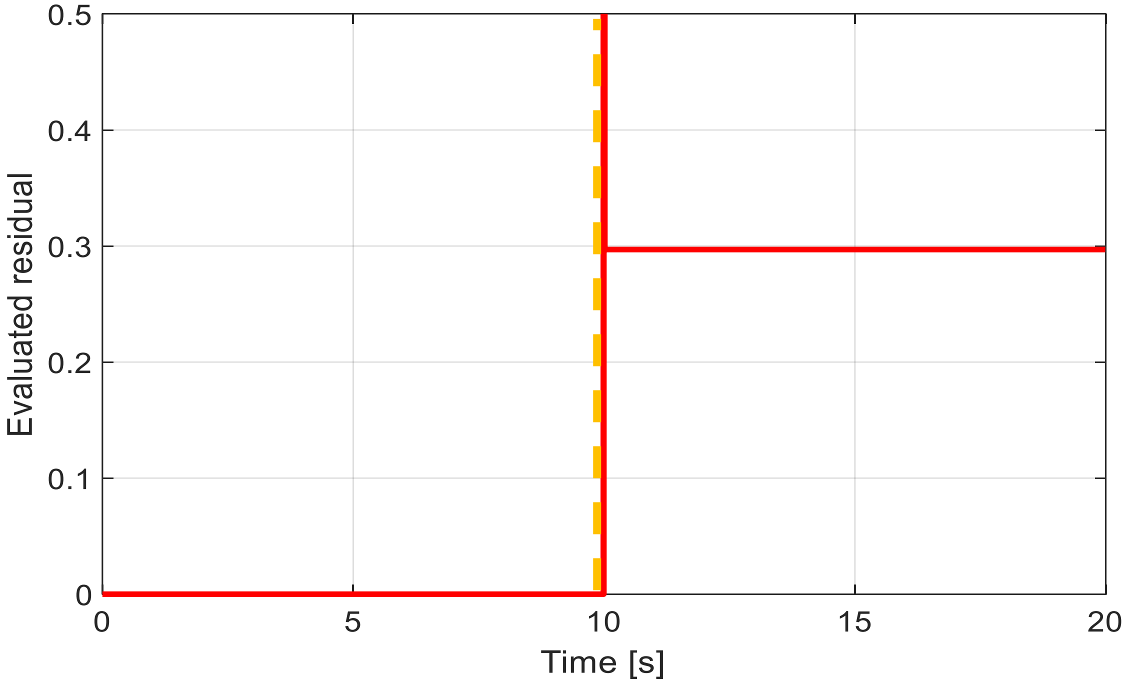
In against, the presented form of $r(t)$ is **not ideal for an on-line implementation**, since not only the actual, but also the past measurements and input data need to be recorded

Parity space: DC motor example

Suppose there is a **STEP fault**, $f(t) = 20 \cdot \mathrm{step}(t)$ on the **speed sensor** at time $t = 10$

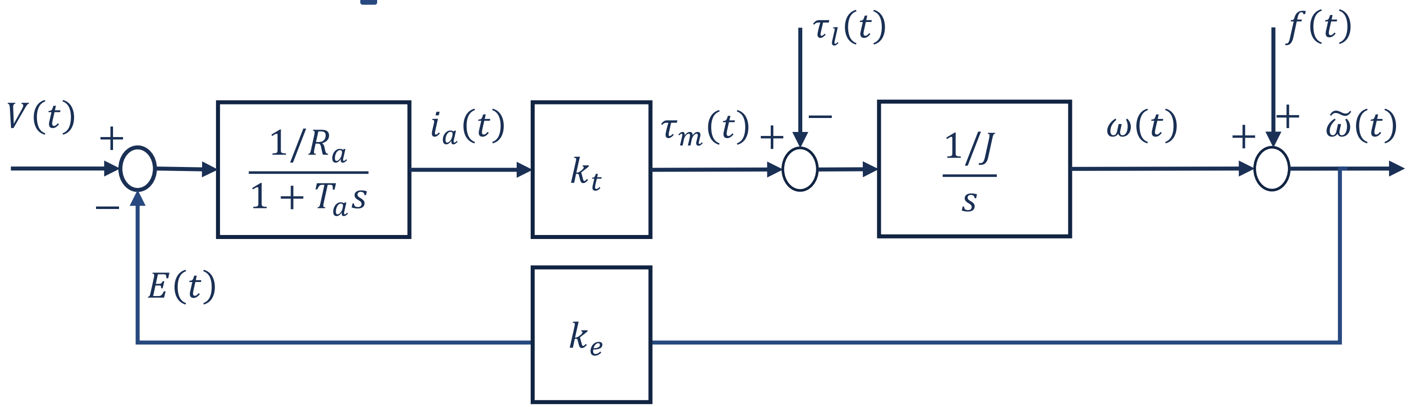


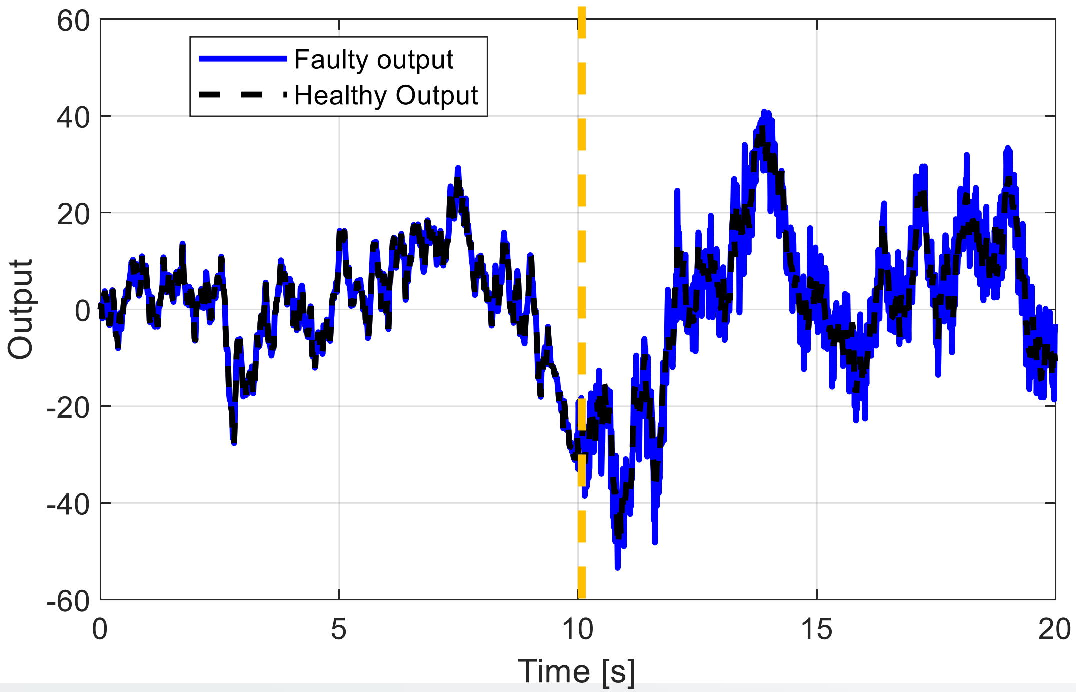


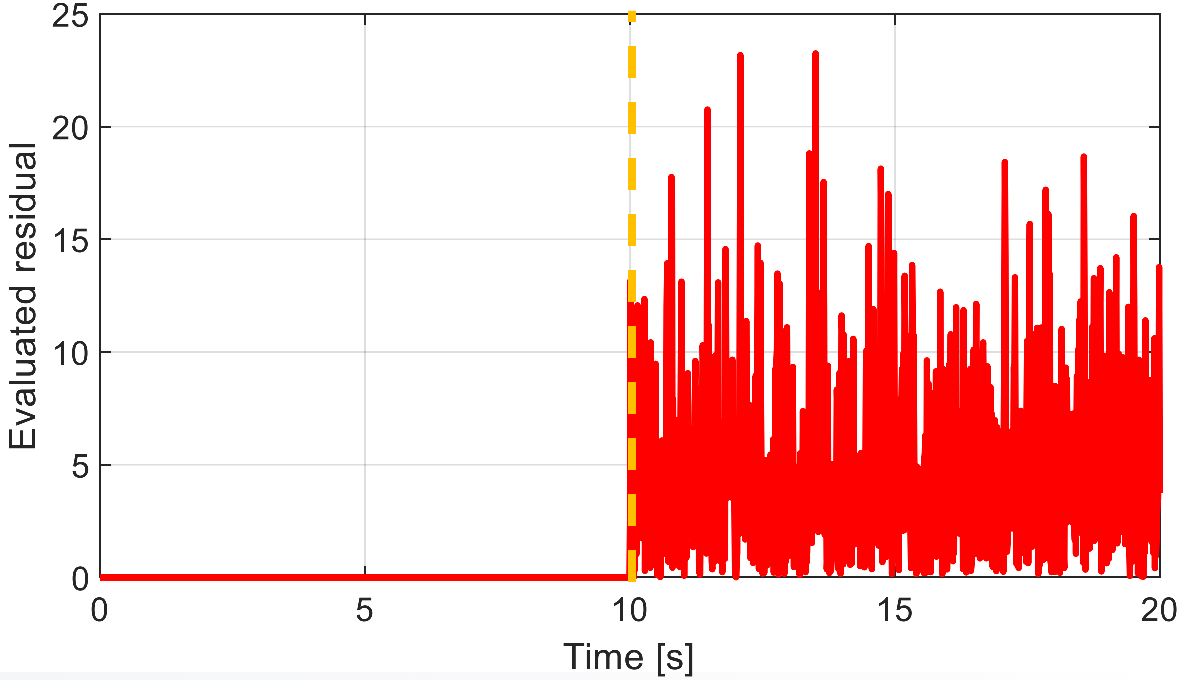


Parity space: DC motor example

Suppose there is a **NOISE fault**, $f(t) = WN(0, \lambda^2)$ with $\mathrm{SNR} = 10$ on the **speed sensor** at time $t = 10$







Parity space: input decoupling

For the fault detection purpose, an ideal residual generation would be a residual signal that **only depends on the faults** to be detected and is simultaneously **independent of the disturbances**

Recall the form of the parity space based residual generator, $\boldsymbol{v}_s \in P_s$:

$$r(t) = \boldsymbol{v}_s^T \cdot \left(H_{d,s}\boldsymbol{d}_s(t) + H_{f,s}\boldsymbol{f}_s(t) \right)$$

Thus, a **residual decoupled** from $\boldsymbol{d}_s(t)$ is delivered if and only if there exists $\boldsymbol{v}_s \in P_s$ s.t.

$\boldsymbol{v}_s^T H_{f,s} \neq \mathbf{0}$ **and** $\boldsymbol{v}_s^T H_{d,s} = \mathbf{0}$

Parity space: input decoupling

The previous condition can be equivalently restated as (with $\Delta \neq \mathbf{0} \in \mathbb{R}^{1\times m_f(s+1)}$)

$$\underbrace{\boldsymbol{v}_s^T}_{1\times p(s+1)} \cdot [\underbrace{H_{f,s}}_{p(s+1)\times m_f(s+1)} \quad \underbrace{H_{o,s}}_{p(s+1)\times n} \quad \underbrace{H_{d,s}}_{p(s+1)\times m_d(s+1)}] = [\underbrace{\Delta}_{1\times m_f(s+1)} \quad \mathbf{0} \quad \mathbf{0}]$$

So, the residual $r(t)$ is decoupled from $\boldsymbol{d}_s(t)$ **if and only if**

$$\text{rank}([H_{f,s} \quad H_{o,s} \quad H_{d,s}]) > \text{rank}([H_{o,s} \quad H_{d,s}])$$

Generally, this condition is fulfilled is there is a number of output measurements which is greater than the number of unobserved inputs, i.e. $p > (m_d + m_f)$

i.e. the colums of the matrix $H_{f,s}$ **are not linear combinations** of the columns of the matrix $[H_{o,s} \quad H_{d,s}]$

Parity space: input decoupling

The **parity space based residual** design algorithm is therefore:

1. Solve, for some s such that the input decoupling condition holds, the problem

$$\boldsymbol{v}_s^T H_{f,s} \neq \mathbf{0} \qquad \textbf{and} \qquad \boldsymbol{v}_s^T [H_{o,s} \quad H_{d,s}] = \mathbf{0}$$

2. Construct the residual generator as follows

$$r(t) = \boldsymbol{v}_s^T \cdot \left(\boldsymbol{y}_s(t) - H_{u,s}\, \boldsymbol{u}_s(t) \right)$$

This leads to a residue $r(t) = \boldsymbol{v}_s^T \cdot \left(H_{o,s}\boldsymbol{x}(t-s) + H_{d,s}\boldsymbol{d}_s(t) + H_{f,s}\boldsymbol{f}_s(t) \right) = H_{f,s}\boldsymbol{f}_s(t)$

Definition of B_f, D_f and B_d, D_d

Recall our model for a system with **additive faults and disturbances**

$$
\begin{cases}
\underset{n\times 1}{\boldsymbol{x}(t+1)} = \underset{n\times n}{A} \cdot \boldsymbol{x}(t) + \underset{n\times m_u}{B} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{n\times m_d}{B_d} \cdot \underset{m_d\times 1}{\boldsymbol{d}(t)} + \underset{n\times m_f}{B_f} \cdot \underset{m_f\times 1}{\boldsymbol{f}(t)} \\
\underset{p\times 1}{\boldsymbol{y}(t)} = \underset{p\times n}{C} \cdot \boldsymbol{x}(t) + \underset{p\times m_u}{D} \cdot \boldsymbol{u}(t) + \underset{p\times m_d}{D_d} \cdot \boldsymbol{d}(t) + \underset{p\times m_f}{D_f} \cdot \boldsymbol{f}(t)
\end{cases}
$$

In order to define B_d, D_d, B_f, D_d, it is useful (but not mandatory) to **start from the transfer functions** $\boldsymbol{G}_{yd}(z)$ and $\boldsymbol{G}_{yf}(z)$, that describe the effect of disturbances and faults on output signals, respectively. Then a **state space realization** can be performed to get B_d, D_d, B_f, D_d

Definition of B_f, D_f and B_d, D_d

We can express the relations of all the external inputs to the output signals as

$$\underset{p\times 1}{\boldsymbol{y}(t)} = \underset{p\times m_u}{\boldsymbol{G}_{yu}(z)} \cdot \boldsymbol{u}(t) + \underset{p\times m_d}{\boldsymbol{G}_{yd}(z)} \cdot \boldsymbol{d}(t) + \underset{p\times m_f}{\boldsymbol{G}_{yf}} \cdot \boldsymbol{f}(t)$$

Realization

$$\begin{cases} \underset{n\times 1}{\boldsymbol{x}(t+1)} = \underset{n\times n}{A} \cdot \boldsymbol{x}(t) + \underset{n\times m_u}{B} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{n\times m_d}{B_d} \cdot \underset{m_d\times 1}{\boldsymbol{d}(t)} + \underset{n\times m_f}{B_f} \cdot \underset{m_f\times 1}{\boldsymbol{f}(t)} \\ \underset{p\times 1}{\boldsymbol{y}(t)} = \underset{p\times n}{C} \cdot \boldsymbol{x}(t) + \underset{p\times m_u}{D} \cdot \boldsymbol{u}(t) + \underset{p\times m_d}{D_d} \cdot \boldsymbol{d}(t) + \underset{p\times m_f}{D_f} \cdot \boldsymbol{f}(t) \end{cases}$$

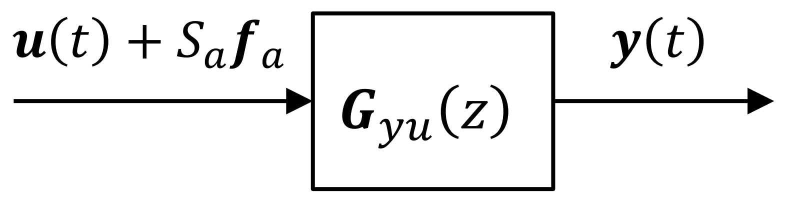
The transfer function $\boldsymbol{G}_{yd}$ **is defined by the physics** of the problem

Definition of B_f, D_f and B_d, D_d

The transfer function $\boldsymbol{G}_{yf}(z)$ **is implicitly defined** by how we choose to **model the fault**

Actuator faults $\boldsymbol{f}_a$

Modelling can be done by replacing $\boldsymbol{u}(t)$ by a perturbed input $\boldsymbol{u}(t) + S_a \boldsymbol{f}_a(t)$, with $S_a \in \mathbb{R}^{m_u \times m_f}$ a **fault distribution matrix**



$$Y(z) = \boldsymbol{G}_{yu}(z)U(z) + \boldsymbol{G}_{yu}(z)S_a \cdot F_a(z)$$

$$\boxed{\boldsymbol{G}_{yf}(z) = \boldsymbol{G}_{yu}(z)S_a}$$

Example

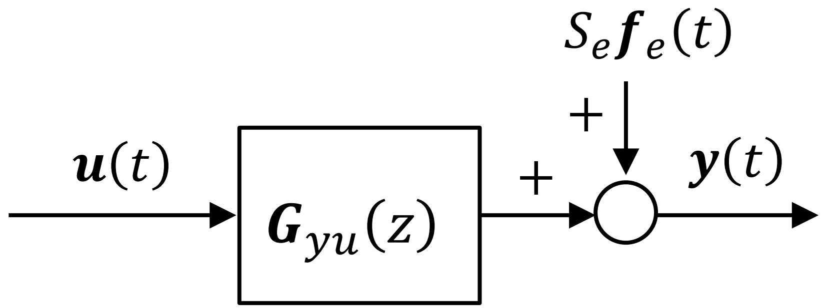
$$\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{S_a} \underbrace{\begin{bmatrix} f_{a_1}(t) \\ f_{a_2}(t) \end{bmatrix}}_{\boldsymbol{f}_a(t)}$$

Definition of B_f, D_f and B_d, D_d

The transfer function $\boldsymbol{G}_{yf}(z)$ **is implicitly defined** by how we choose to **model the fault**

Sensor faults $\boldsymbol{f}_e$

Modelling can be done by replacing $\boldsymbol{y}(t)$ by a perturbed output $\boldsymbol{y}(t) + S_e \boldsymbol{f}_e(t)$, with $S_e \in \mathbb{R}^{p \times m_f}$ a **fault distribution matrix**



$$Y(z) = \boldsymbol{G}_{yu}(z)U(z) + S_e \cdot F_e(z)$$

$$\boxed{\boldsymbol{G}_{yf}(z) = S_e}$$

Example

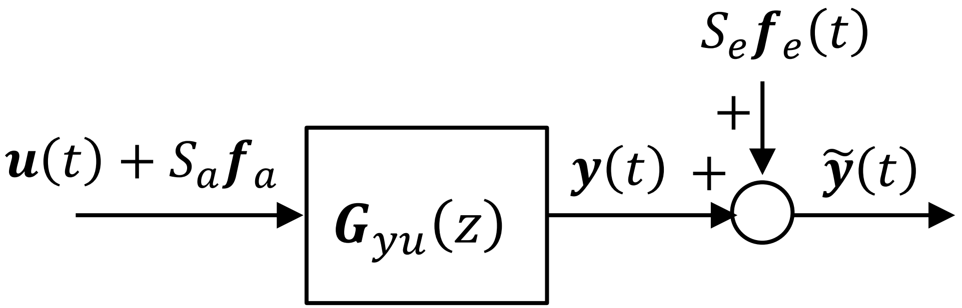
$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\boldsymbol{y}(t)} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{S_e} \underbrace{\begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \end{bmatrix}}_{\boldsymbol{f}_e(t)}$$

Definition of B_f, D_f and B_d, D_d

The transfer function $\boldsymbol{G}_{yf}(z)$ **is implicitly defined** by how we choose to **model the fault**

Actuator faults $\boldsymbol{f}_a$ + Sensor faults $\boldsymbol{f}_e$

The modeling is the union of the previous cases



$$Y(z) = \boldsymbol{G}_{yu}(z)U(z) + \boldsymbol{G}_{yu}S_a \cdot F_a(z)$$

$$\tilde{Y}(z) = Y(z) + S_eF_e(z) \quad \Rightarrow \quad \tilde{Y}(z) = Y(z) + S_eF_e(z) = \boldsymbol{G}_{yu}(z)U(z) + [\boldsymbol{G}_{yu}(z)S_a \quad S_e]\begin{bmatrix} F_a(z) \\ F_e(z) \end{bmatrix}$$

$$\boxed{\boldsymbol{G}_{yf}(z) = [\boldsymbol{G}_{yu}(z)S_a \quad S_e]}$$

$$\boldsymbol{f}(t) = \begin{bmatrix} \boldsymbol{f}_a(t) \\ \boldsymbol{f}_e(t) \end{bmatrix}$$

Faults vector

Parity space: DC motor example input decoupling

We want to build a parity vector **decoupled from the load disturbance** $\tau_l(t)$. In order to be able to do this, we suppose that a **2nd measurement output** is available, e.g. we measure the motor speed $\omega(t)$ with **another sensor** (with different gain)

Furthermore, suppose that we have a sensor fault, s.t. $Y(z) = \boldsymbol{G}_{yu}(z)U(z) + S_e \cdot F_e(z)$, with $\underset{\substack{p\times m_f \\ 2\times 1}}{S_e = \boldsymbol{G}_{yf}(z)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,i.e. the fault acts only on the 1° output (1° speed sensor)

$$\underset{\substack{p\times m_u \\ 2\times 1}}{\boldsymbol{G}_{yu}(z)} = \begin{bmatrix} \dfrac{1.184\,z + 1.184}{z^2 - 0.9852\,z + 2.943\cdot 10^{-5}}\cdot 1 \\ \dfrac{1.184\,z + 1.184}{z^2 - 0.9852\,z + 2.943\cdot 10^{-5}}\cdot 0.99 \end{bmatrix} \qquad \underset{\substack{p\times m_d \\ 2\times 1}}{\boldsymbol{G}_{yd}(z)} = \begin{bmatrix} \dfrac{-123.6z + 0.003637}{z^2 - 0.9852\,z + 2.943\cdot 10^{-5}}\cdot 1 \\ \dfrac{-123.6z + 0.003637}{z^2 - 0.9852\,z + 2.943\cdot 10^{-5}}\cdot 0.99 \end{bmatrix}$$

Parity space: DC motor example input decoupling

A state space realization of $\boldsymbol{G}_{yu}(z)$ leads to

$$\underset{\substack{n \times n \\ 2 \times 2}}{A} = \begin{bmatrix} 0.9851 & 0.0021 \\ 0.0019 & 0 \end{bmatrix} \quad \underset{\substack{n \times m_u \\ 2 \times 1}}{B} = \begin{bmatrix} -0.3087 \\ -0.1275 \end{bmatrix} \quad \underset{\substack{p \times n \\ 1 \times 2}}{C} = \begin{bmatrix} -7.7413 & 18.95 \end{bmatrix} \quad \underset{\substack{p \times m_u \\ 2 \times 1}}{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\substack{n \times m_d \\ 2 \times 1}}{B_d} = \begin{bmatrix} 15.9988 \\ 0.0315 \end{bmatrix} \quad \underset{\substack{n \times m_f \\ 2 \times 1}}{B_f} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \underset{\substack{p \times m_d \\ 2 \times 1}}{D_d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \underset{\substack{p \times m_f \\ 2 \times 1}}{D_f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underset{\substack{p(s+1) \times n \\ 2(2+1) \times 2}}{H_{o,s}} = \begin{bmatrix} -7.7413 & 18.9468 \\ -7.6639 & 9.3634 \\ -7.6079 & -0.0156 \\ -7.5319 & -0.0154 \\ -7.4950 & -0.0156 \\ -7.4200 & -0.0154 \end{bmatrix} \quad H_{u,s} = \underset{\substack{p(s+1) \times m_u(s+1) \\ 2(2+1) \times 1(2+1)}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.1842 & 0 & 0 \\ 1.1723 & 0 & 0 \\ 2.3508 & 1.1842 & 0 \\ 2.3273 & 1.1723 & 0 \end{bmatrix}} \quad H_{f,s} = \underset{\substack{p(s+1) \times m_f(s+1) \\ 2(2+1) \times 1(2+1)}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \quad H_{d,s} = \underset{\substack{p(s+1) \times m_d(s+1) \\ 2(2+1) \times 1(2+1)}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -123.5527 & 0 & 0 \\ -122.3172 & 0 & 0 \\ -121.7180 & -123.5527 & 0 \\ -120.5008 & -122.3172 & 0 \end{bmatrix}}$$

Parity space: DC motor example input decoupling

We want to solve for $\boldsymbol{v}_s$ the following problem:

$$\boldsymbol{v}_s^T H_{f,s} \neq \boldsymbol{0} \qquad \textbf{and} \qquad \boldsymbol{v}_s^T [H_{o,s} \quad H_{d,s}] = \boldsymbol{0}$$

We can recast it in the standard form for solving linear systems: $[H_{o,s} \quad H_{d,s}]^T \boldsymbol{v}_s = \boldsymbol{0}$

This system is homogeneous admits always the particular solution $\boldsymbol{v}_s^o = \boldsymbol{0}$. If the condition $\mathrm{rank}([H_{f,s} \quad H_{o,s} \quad H_{d,s}]) > \mathrm{rank}([H_{o,s} \quad H_{d,s}])$ holds, one way to obtain a nicer solution is to add a solution from the **nullspace** of $[H_{o,s} \quad H_{d,s}]^T$

$$\boldsymbol{v}_s^{\mathrm{dec}} = \boldsymbol{v}_s^0 + \mathrm{null}([H_{o,s} \quad H_{d,s}]^T) \cdot [\alpha_1 \dots a_{ns}]^T$$

- $a \in \mathbb{R}$
- n_s: dimension of null space

Check a-posteriori if $\boldsymbol{v}_s^T H_{f,s} \neq \boldsymbol{0}$

Parity space: DC motor example input decoupling

Always check the solution!

$$\left(\boldsymbol{v}_s^{\text{dec}}\right)^T [H_{o,s} \quad H_{d,s}] =$$

$$= [-0.9900 \quad 1 \quad -1.9800 \quad 2 \quad -2.9700 \quad 3] \cdot \begin{bmatrix} -7.7413 & 18.9468 & 0 & 0 & 0 \\ -7.6639 & 9.3634 & 0 & 0 & 0 \\ -7.6079 & -0.0156 & -123.5527 & 0 & 0 \\ -7.5319 & -0.0154 & -122.3172 & 0 & 0 \\ -7.4950 & -0.0156 & -121.7180 & -123.5527 & 0 \\ -7.4200 & -0.0154 & -120.5008 & -122.3172 & 0 \end{bmatrix}$$

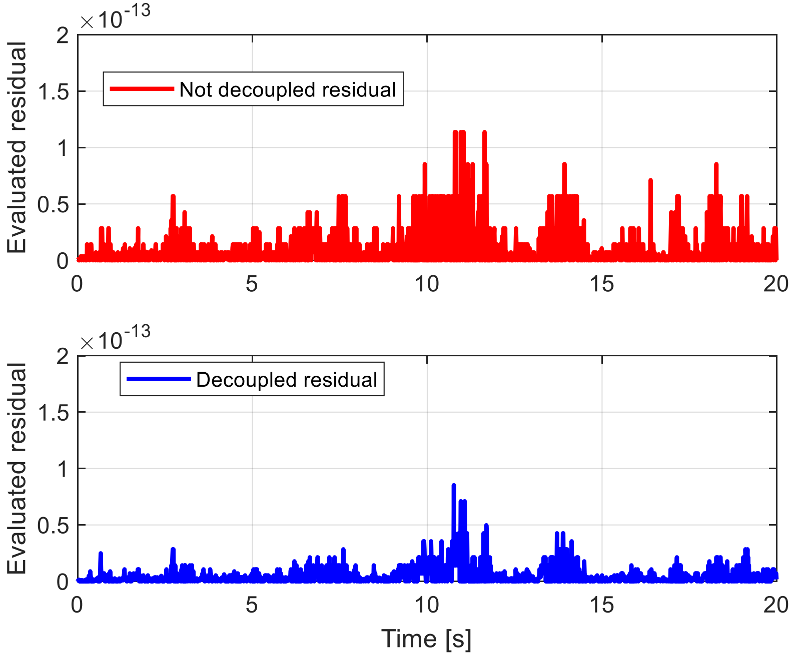
$$= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$ **CORRECT! ☺**

$$\left(\boldsymbol{v}_s^{\text{dec}}\right)^T H_{f,s} = [-0.99 \quad 1 \quad -1.98 \quad 2 \quad -2.97 \quad 3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = [-0.99 \quad -1.98 \quad -2.97] \neq 0$$ **OK! ☺**

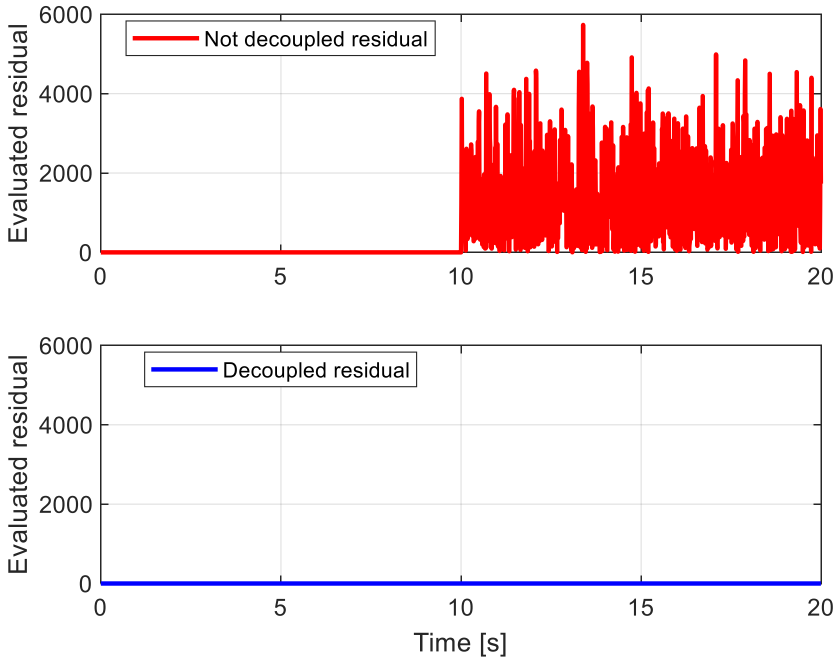
Parity space: DC motor example input decoupling

We can now compare the results of the **decoupled parity vector** v_s^{dec}, with respect to the **not decoupled parity vector** v_s

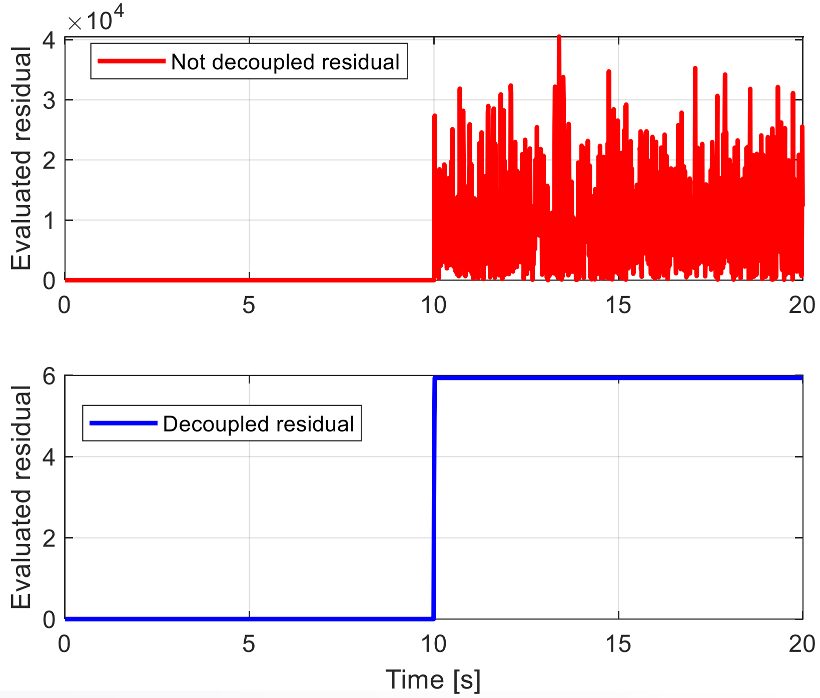
Residual with $d(t) = f(t) = 0$



Residual with $d(t) = WN(0, \lambda^2)$ λ^2 s.t. SNR $= 1$, $f(t) = 0$



Residual with $d(t) = WN(0, \lambda^2)$ λ^2 s.t. SNR $= 1$, $f(t) = 1 \cdot \mathrm{step}(t)$



Outline

1. Schematic of the approach
2. Dynamical systems
3. Parity space approach
4. **Diagnostic observer**
5. Application to EMA fault detection

Diagnostic observer

A **disadvantage** of the parity approach method is that it requires to **store a sample** of past inputs and outputs at time $t-1, \dots, t-s$, in order to compute the residual signal at the current time t. The implementation is **non-recursive**

It is possible to employ a different residual generator scheme, known as diagnostic observer, by **using the designed parity vector $\boldsymbol{v}_s$**

This allows to employ a **recursive implementation**, s.t. $r(t)$ is computed from $r(t-1)$

A common strategy based on **«parity space design, observer-based implementation»** is vastly used

Diagnostic observer

A **diagnostic observer** can be formulated as the following **dynamical system**

$$\begin{cases} \underset{s\times 1}{\boldsymbol{m}(t+1)} = \underset{s\times s}{G} \cdot \underset{s\times 1}{\boldsymbol{m}(t)} + \underset{s\times m_u}{H} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{s\times p}{L} \cdot \underset{p\times 1}{\boldsymbol{y}(t)} \\ \underset{1\times 1}{r(t)} = -\underset{1\times s}{w} \cdot \underset{s\times 1}{\boldsymbol{m}(t)} - \underset{1\times m_u}{q} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{1\times p}{v} \cdot \underset{p\times 1}{\boldsymbol{y}(t)} \end{cases}$$

We can interpret the quantity $-w \cdot \boldsymbol{m}(t) - q \cdot \boldsymbol{u}(t)$ as an **estimate** for $v\boldsymbol{y}(t)$. In this way, the residual signal reads as $r(t) = \hat{y}(t) - vy(t)$. Notice how $\hat{y}(t)$ depends on $\boldsymbol{m}(t)$, which in turn depends on $y(t-1)$.

For this reason, observer approaches are known as **closed-loop approaches.** This give them **more robustness** with respect to **modeling errors**

Diagnostic observer

Let $\boldsymbol{v}_s^T = \begin{bmatrix} \underbrace{\boldsymbol{v}_{s,0}^T}_{1\times p} & \boldsymbol{v}_{s,1}^T & \cdots & \boldsymbol{v}_{s,s}^T \end{bmatrix}$, be a parity vector, $\boldsymbol{v}_s \in \mathbb{R}^{p(s+1)\times 1}$, $\boldsymbol{v}_s \in P_s$. Then, we have:

$$\underbrace{G}_{s\times s-1} = [G_0 \quad \boldsymbol{g}] \qquad G_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix} \qquad \underbrace{L}_{s\times p} = -\underbrace{\begin{bmatrix} \boldsymbol{v}_{s,0} \\ \vdots \\ \boldsymbol{v}_{s,s-1} \end{bmatrix}}_{s\times p} - \underbrace{\boldsymbol{g}}_{s\times 1} \cdot \underbrace{\boldsymbol{v}_{s,s}}_{1\times p}$$

$$\underbrace{\begin{bmatrix} \underbrace{H}_{s\times m_u} \\ \underbrace{q}_{1\times m_u} \end{bmatrix}}_{s+1\times m_u} = \begin{bmatrix} \overbrace{\boxed{\begin{matrix} \boldsymbol{v}_{s,0} + g_1\boldsymbol{v}_{s,s} & \boldsymbol{v}_{s,1} & \boldsymbol{v}_{s,2} & \cdots & \boldsymbol{v}_{s,s-1} & \boldsymbol{v}_{s,s} \\ \boldsymbol{v}_{s,2} + g_2\boldsymbol{v}_{s,s} & \boldsymbol{v}_{s,2} & \cdots & \cdots & \boldsymbol{v}_{s,s} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \boldsymbol{v}_{s,s} + g_3\boldsymbol{v}_{s,s} & \boldsymbol{v}_{s,s} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{matrix}}}^{s\times p(s+1)} \\ \underbrace{\boxed{\begin{matrix} \boldsymbol{v}_{s,s} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{matrix}}}_{1\times p(s+1)} \end{bmatrix} \underbrace{\begin{bmatrix} D \\ CB \\ CAB \\ \vdots \\ CA^{s-2}B \\ CA^{s-1}B \end{bmatrix}}_{p(s+1)\times m_u}$$

$$\underbrace{\boldsymbol{w}}_{1\times s} = [0 \quad \cdots \quad 1]$$

$$\underbrace{\boldsymbol{v}}_{1\times p} = \boldsymbol{v}_{s,s}$$

Diagnostic observer

$$\begin{cases} \underset{s\times 1}{\boldsymbol{m}(t+1)} = \underset{s\times s}{G} \cdot \underset{s\times 1}{\boldsymbol{m}(t)} + \underset{s\times m_u}{H} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{s\times p}{L} \cdot \underset{p\times 1}{\boldsymbol{y}(t)} \\ \underset{1\times 1}{r(t)} = -\underset{1\times s}{\boldsymbol{w}} \cdot \underset{s\times 1}{\boldsymbol{m}(t)} - \underset{1\times m_u}{\boldsymbol{q}} \cdot \underset{m_u\times 1}{\boldsymbol{u}(t)} + \underset{1\times p}{\boldsymbol{v}} \cdot \underset{p\times 1}{\boldsymbol{y}(t)} \end{cases}$$

The vector $\boldsymbol{g}$ in the matrix $\boldsymbol{G}$ is an **additional degree of freedom** given by the observer formulation. It can be used to assign a **desired dynamic** to the observer

When $\boldsymbol{g} = \mathbf{0}$, then the DO formulation is **analogous** to the parity space design

The obtained residual generator is a **dynamical system** (a **filter)**, that takes as input the **system inputs** $u(t)$ **and outputs** $y(t)$, and get as output the **residual signal** $r(t)$

Diagnostic observer: DC motor

We now apply the DO scheme using the **decoupled parity vector** v_s^{dec} previously found.

Denote with $\boldsymbol{Q}(z)$ **the transfer function of the residual generator** , and let $\boldsymbol{g} = \boldsymbol{0}$

$$\underset{3\times 1}{\boldsymbol{Q}(z)} = \begin{bmatrix} \dfrac{-8.882\cdot 10^{-16}}{z^2} \\ \dfrac{2.97z^2 - 1.98z - 0.99}{z^2} \\ \dfrac{3z^2 - 2z + 1}{z^2} \end{bmatrix}$$

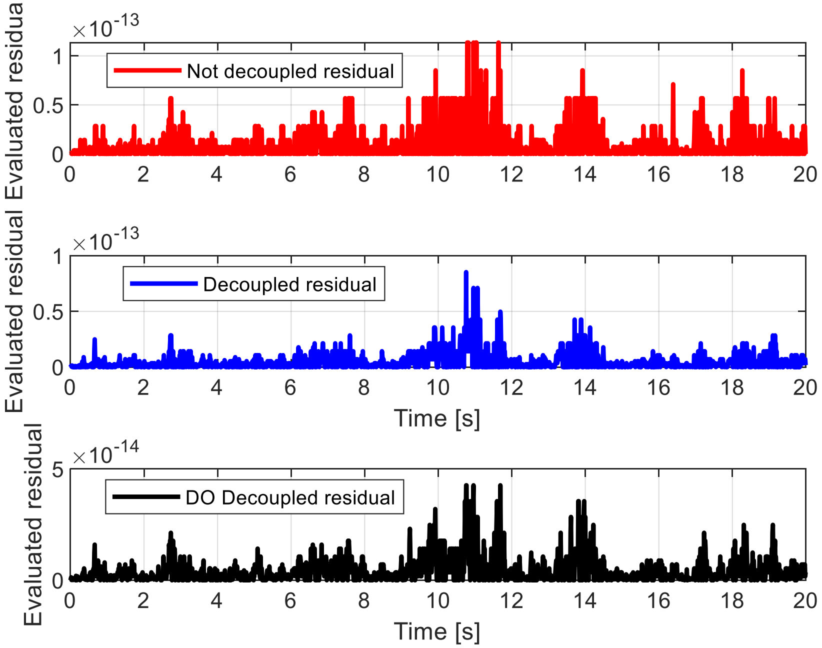
- This filter has **3 inputs** and **1 output**
- 1° input: $u(t)$ - 2nd input: $y_1(t)$ - 3rd input: $y_2(t)$,

  where $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$
- Output: residual signal $r(t)$

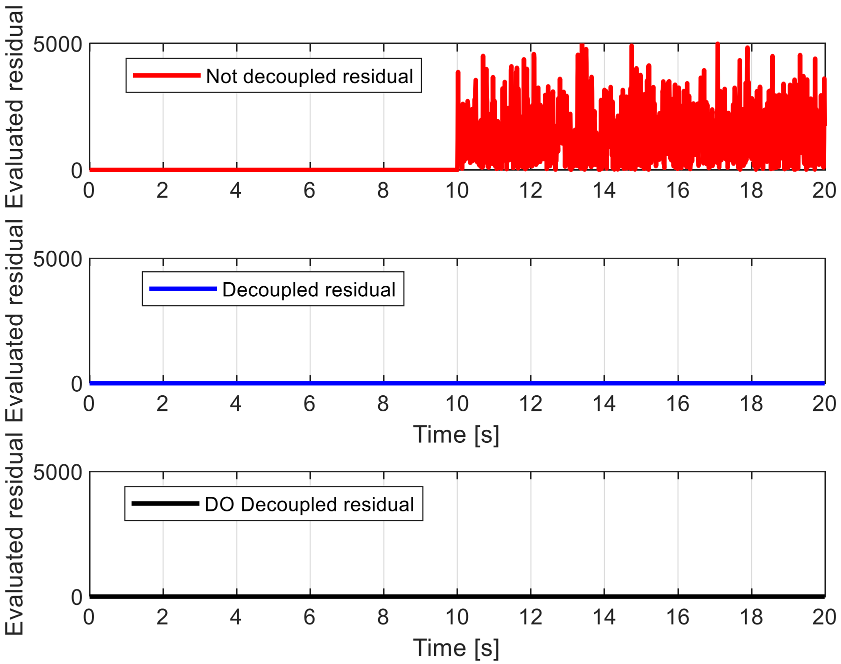
Diagnostic observer: DC motor

Compare the results with the parity base approach

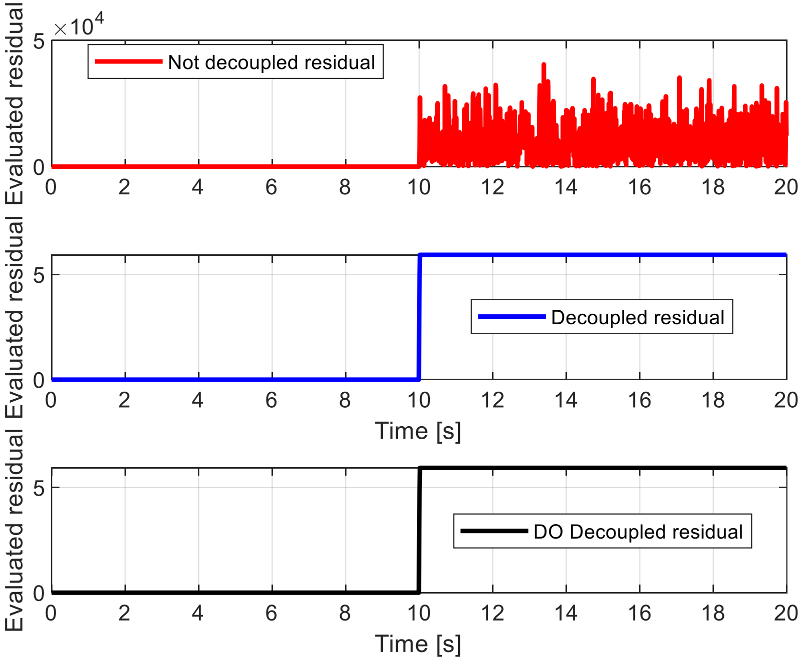
Residual with $d(t) = f(t) = 0$



Residual with $d(t \geq 10) = WN(0, \lambda^2)$
λ^2 s.t. SNR $= 1$, $f(t) = 0$



Residual with $d(t \geq 10) = WN(0, \lambda^2)$
λ^2 s.t. SNR $= 1$, $f(t \geq 10) = 1 \cdot \mathrm{step}(t)$

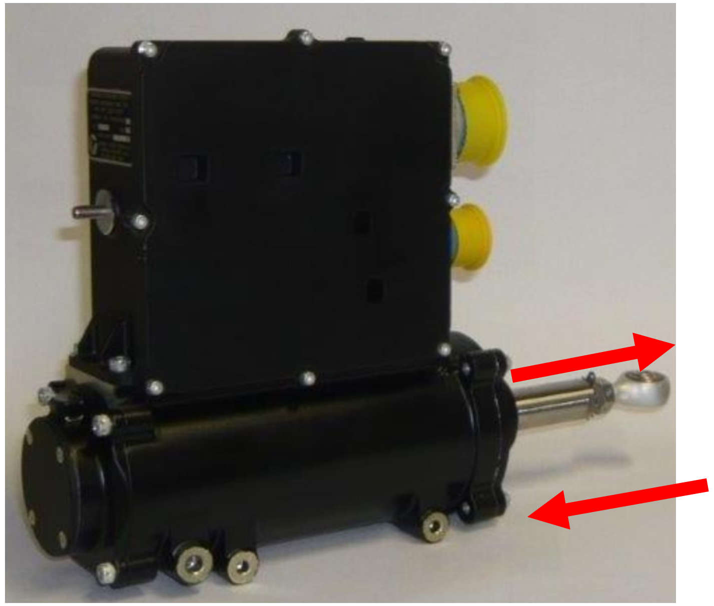


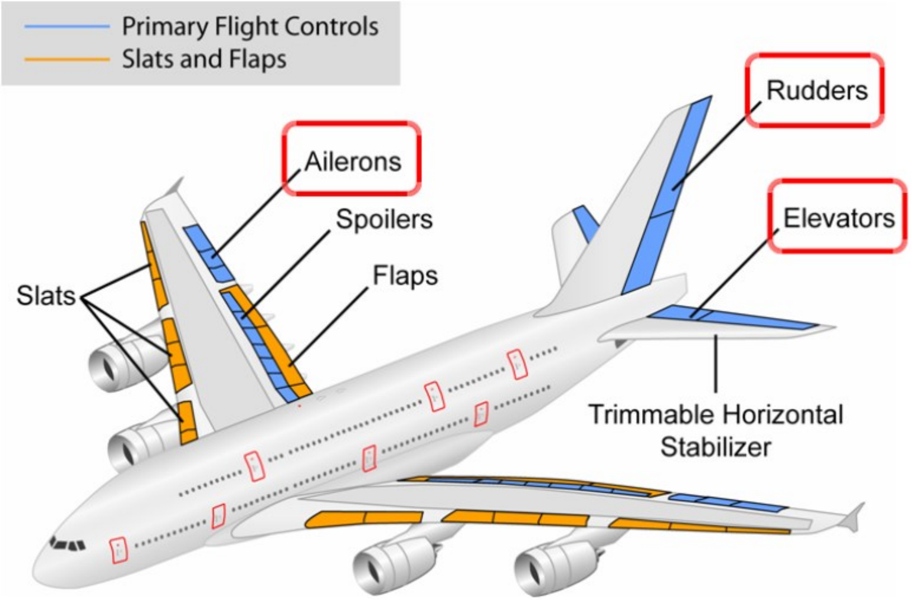
Outline

The REPRISE project

The *Reliable Electromechanical actuator for PRImary SurfacE with health monitoring* is a H2020 project which aims to **design new EMA** for flight actuation systems **with HM capabilities**

This project has received funding from the the Clean Sky 2 Joint Undertaking under the European Union's Horizon 2020 research and innovation programme under grant agreement No 717112 (project acronym: REPRISE).





The REPRISE project

The following activities were carried out to develop a **model-based fault diagnosis** method:

1. Experimental data acquisition (with **endurance** and degradation of **ballscrew** component)

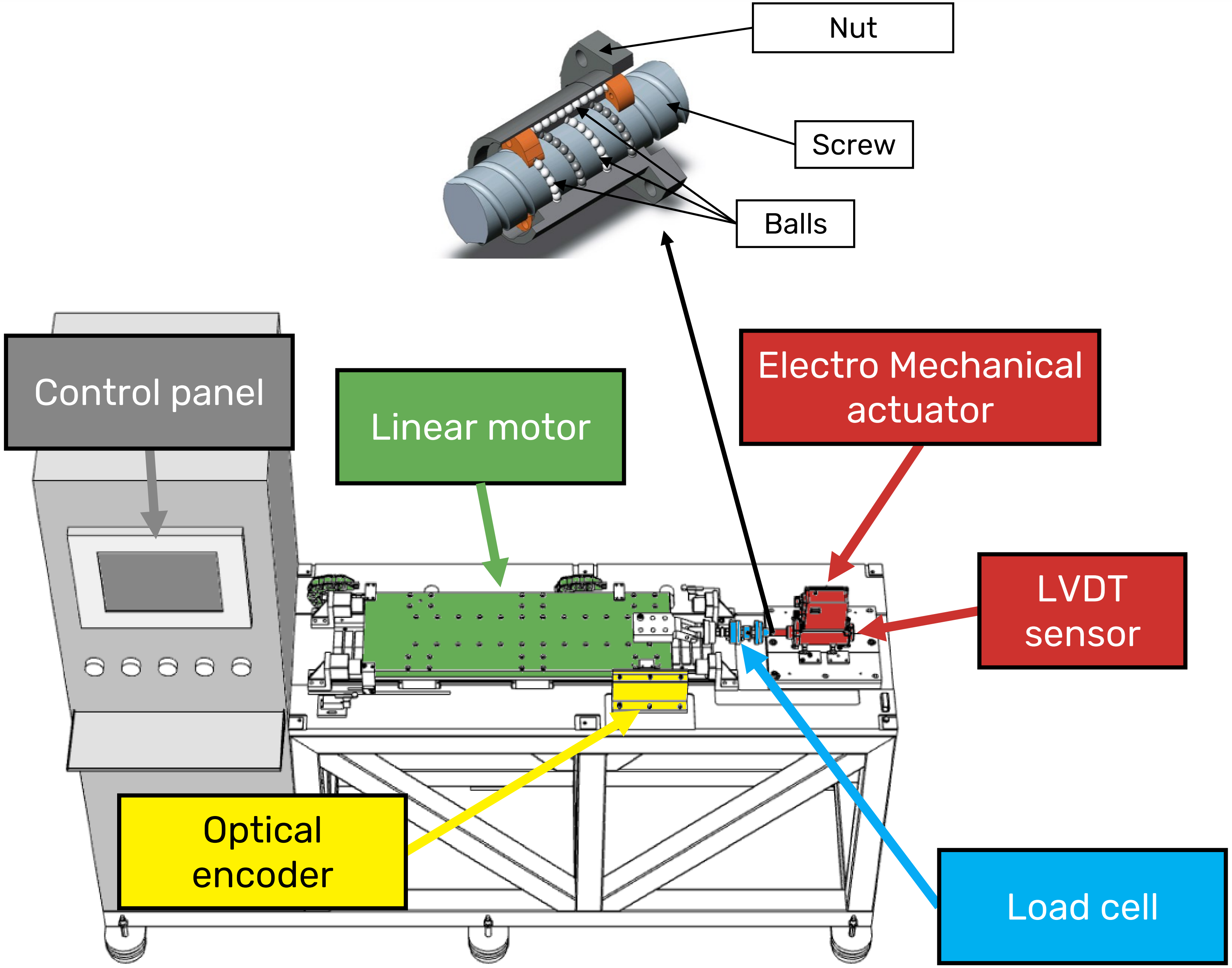
2. Estimation of the EMA **closed-loop** transfer function $G_{yu}(z) = \frac{X(z)}{\bar{X}(z)}$
   - $\bar{x}(t)$: position reference
   - $x(t)$: measured position

3. Linear **residual generator design** based on parity space and diagnostic observer

Test bench layout

Measurements:

1. EMA Phase currents
2. EMA LVDT position
3. Linear motor position
4. EMA Reference position
5. Load cell



Test bench layout

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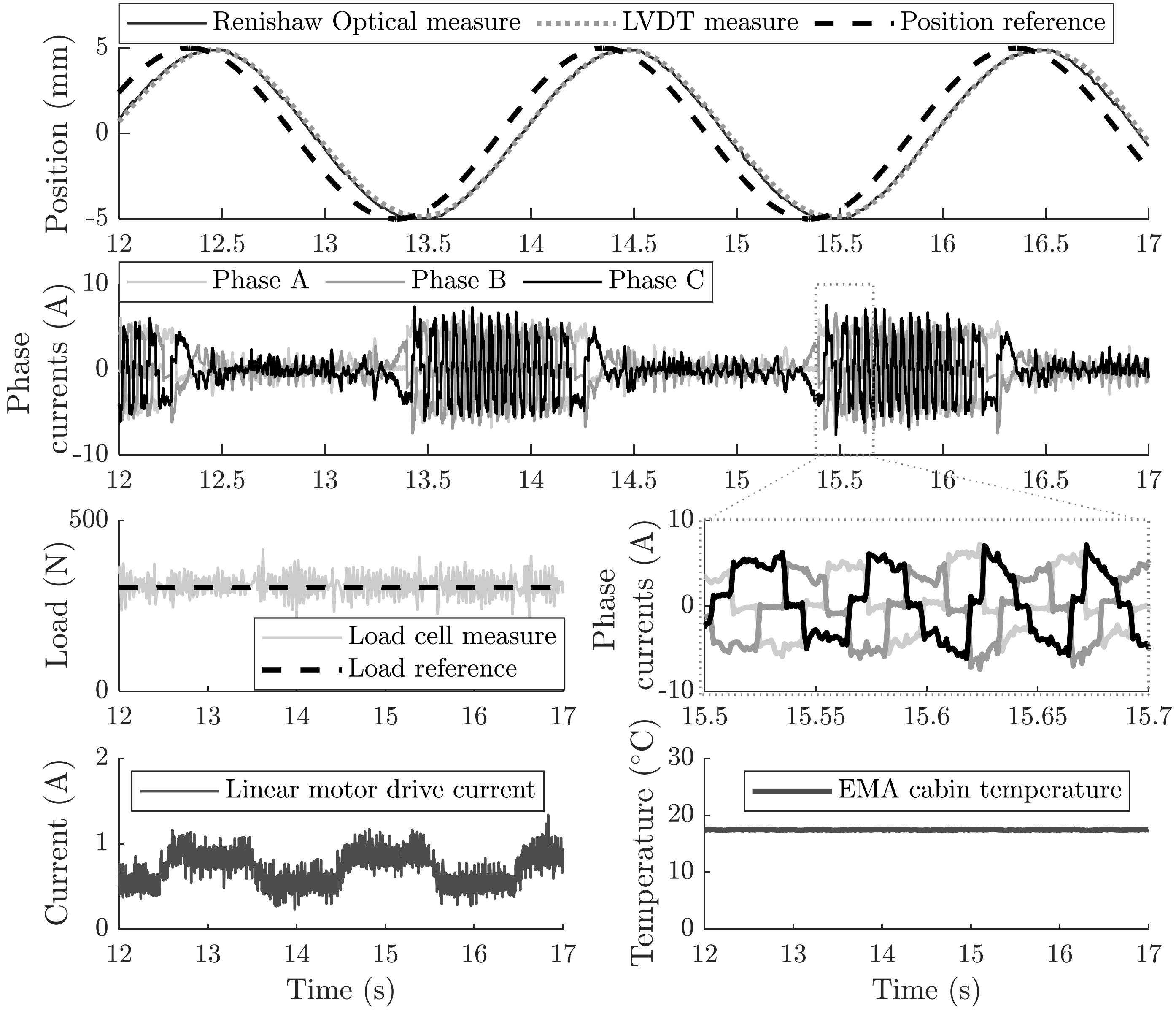
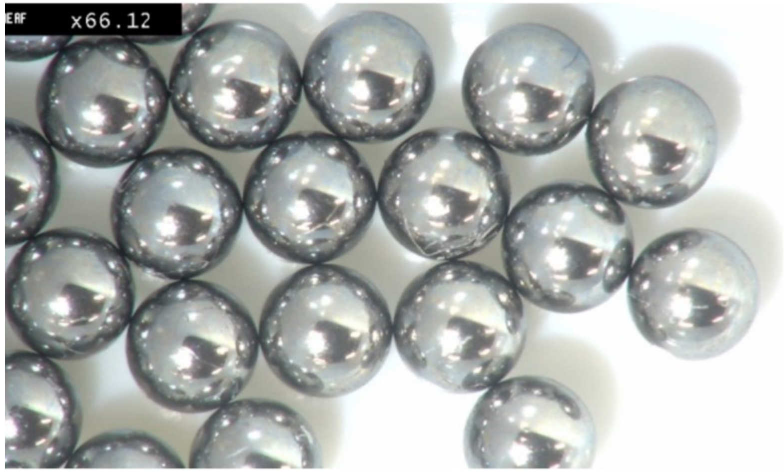
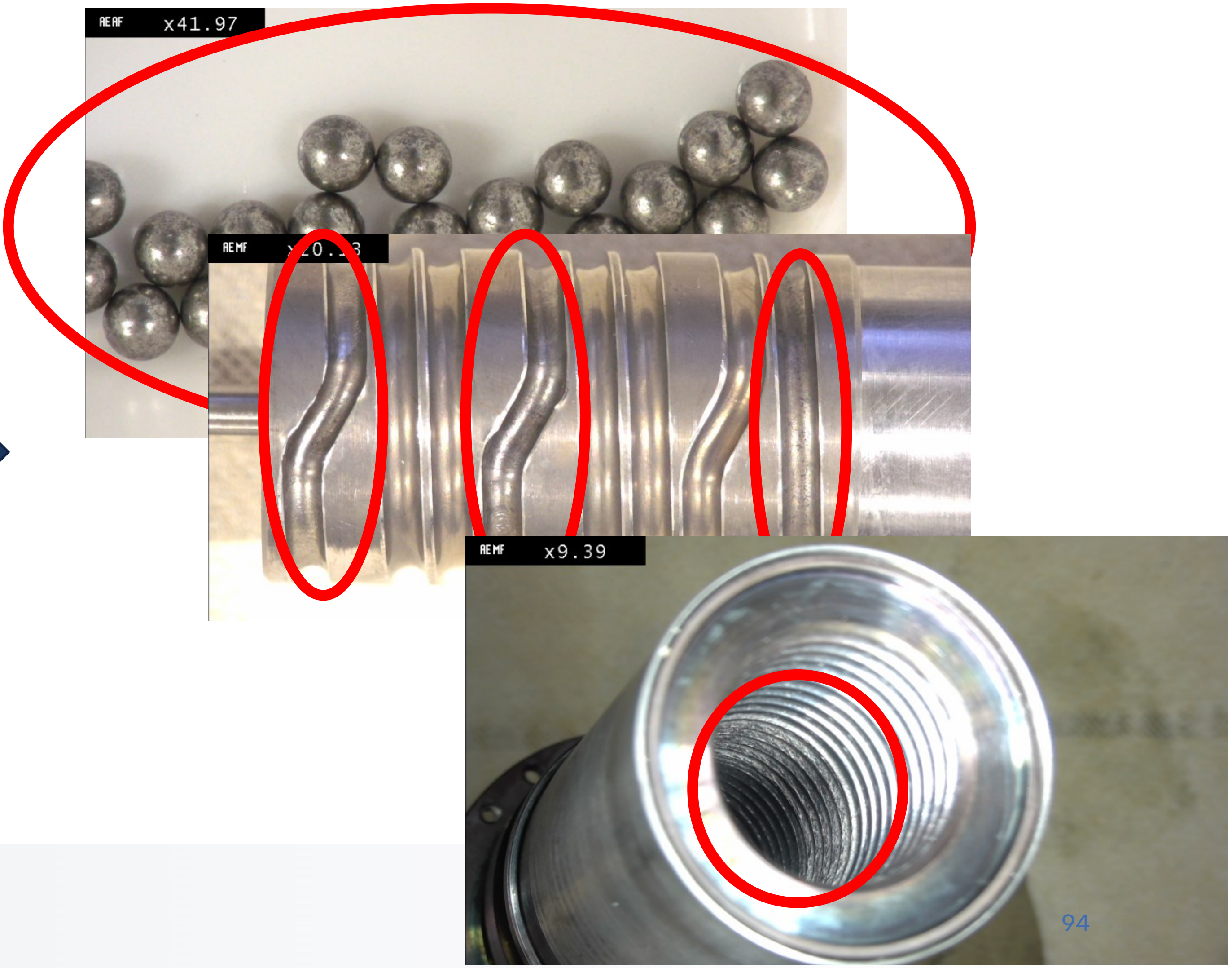


Ballscrew degradation

“Hammered” **balls**, “indented” **recirculation circuits** in ballscrew,“scratched” **nut thread**





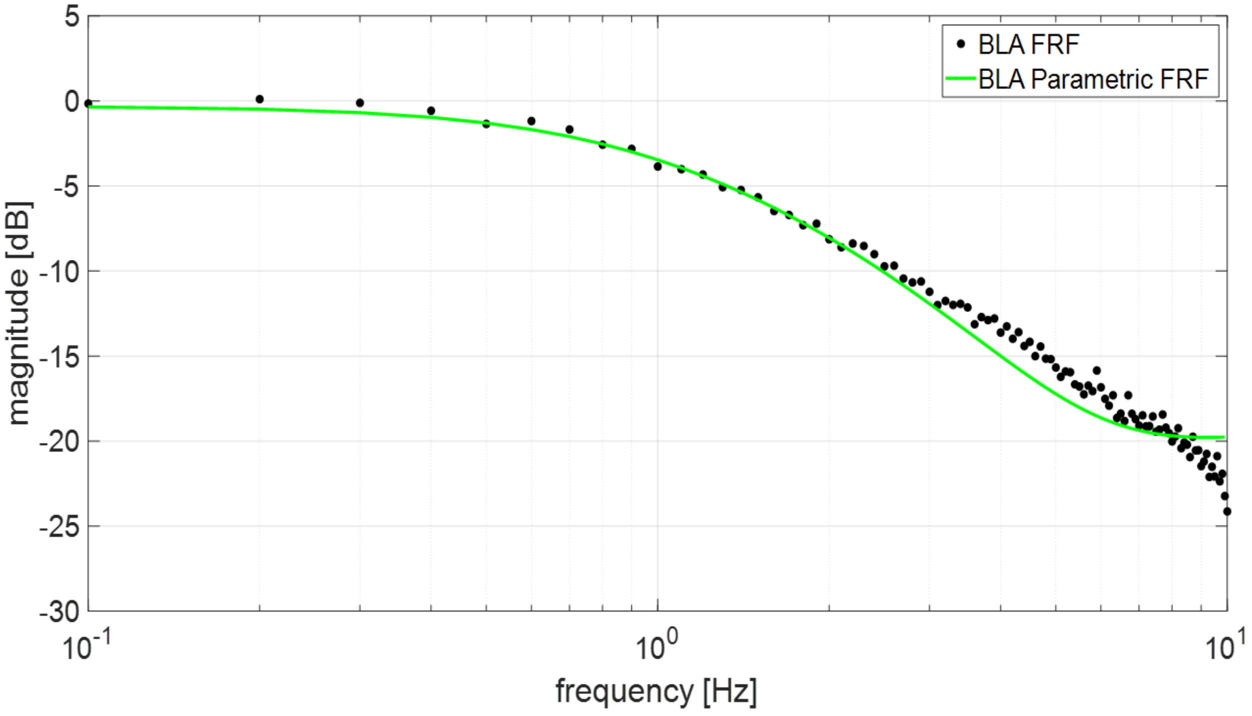
EMA closed-loop model

Identification method:

1. **Multisine** excitation

2. **Nonparametric estimate** of the Best Linear Approximation (BLA)

3. **Parametric estimate** fitting FRF data



$$\frac{X(z)}{\bar{X}(z)} = G(z) = \frac{0.06892 - 0.1732z^{-1} + 0.1266z^{-2}}{1 - 1.624z^{-1} + 0.6467z^{-2}}$$

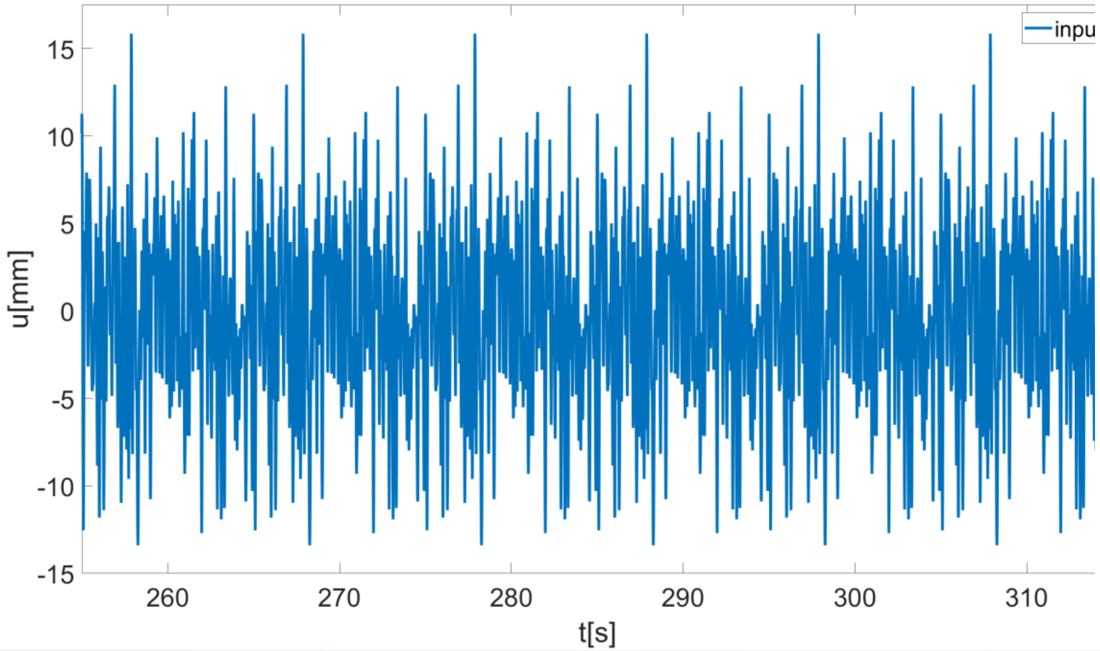
- $\bar{x}(t)$: position reference
- $x(t)$: position measure LVDT

Fault behaviour

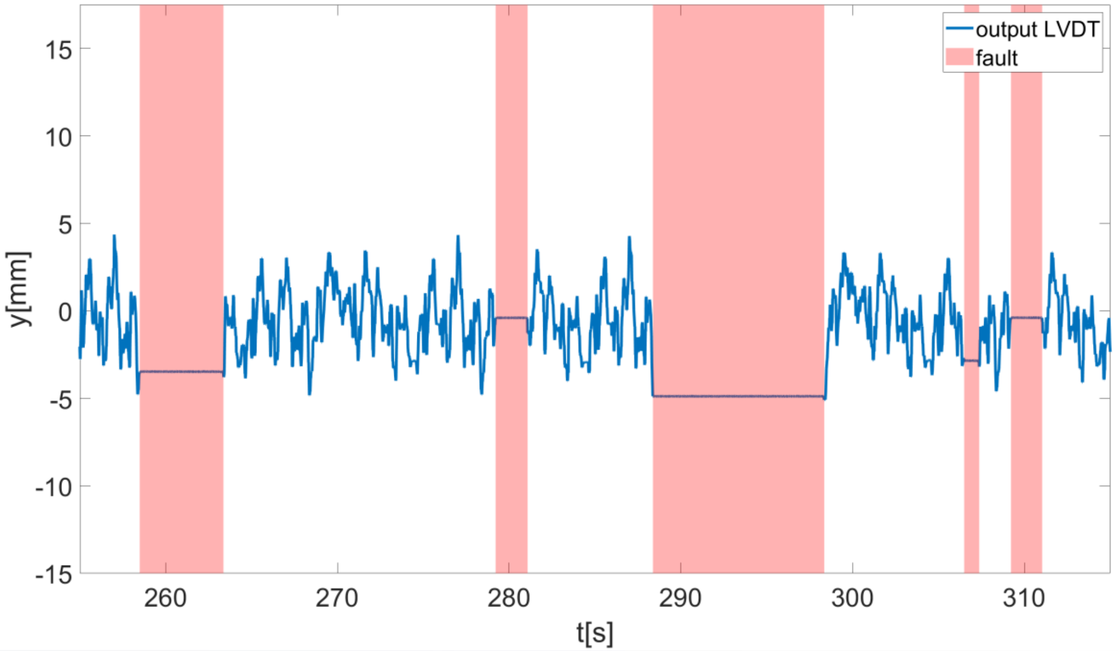
When the degradation took over, the **EMA undergone small-jams** during the operation

The output position signal, as measured by the LVDT sensor, shows **constant value.** We modeled the fault as an **actuator fault**.

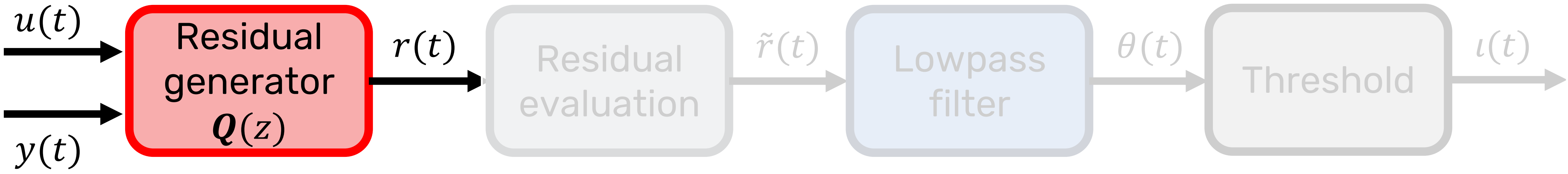
Multisine position reference

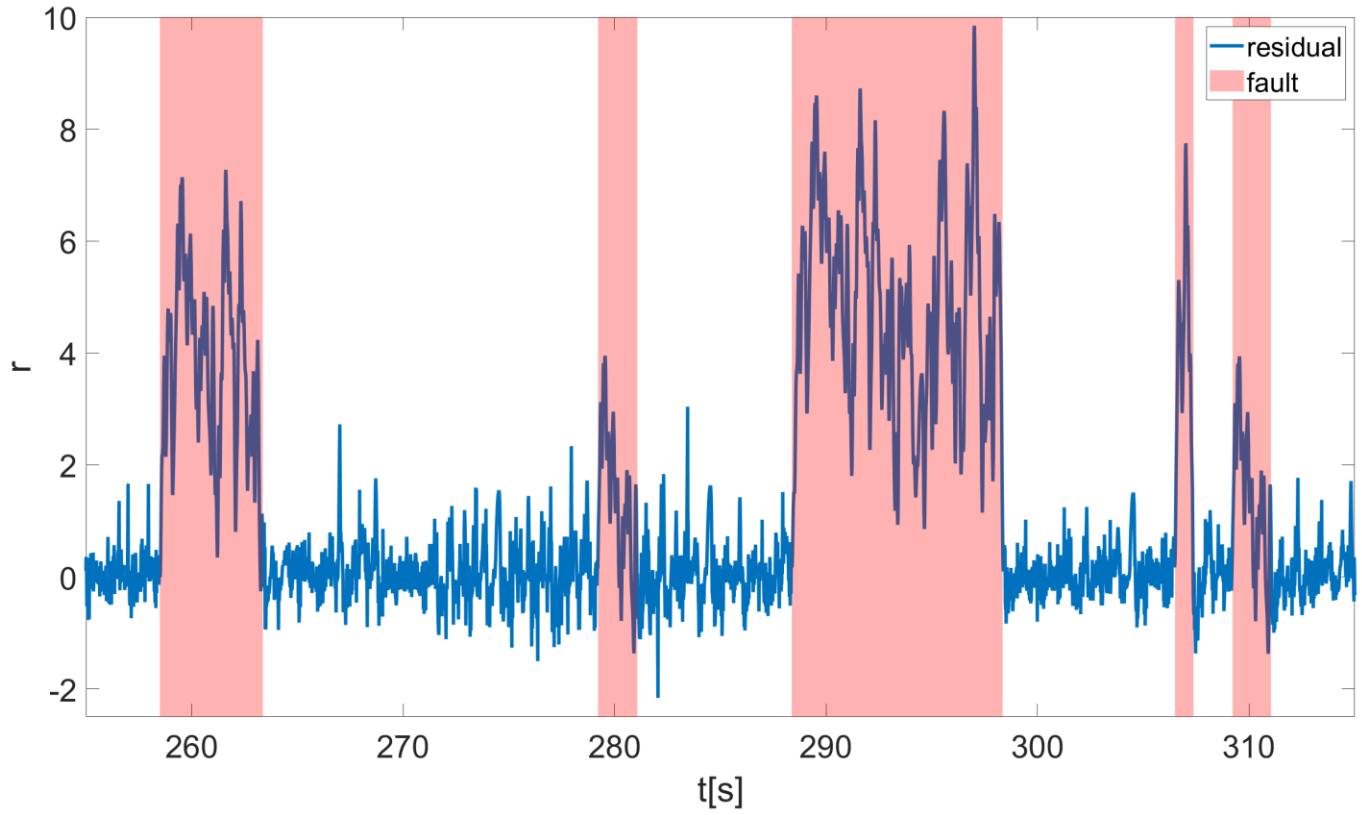


Output position (LVDT)

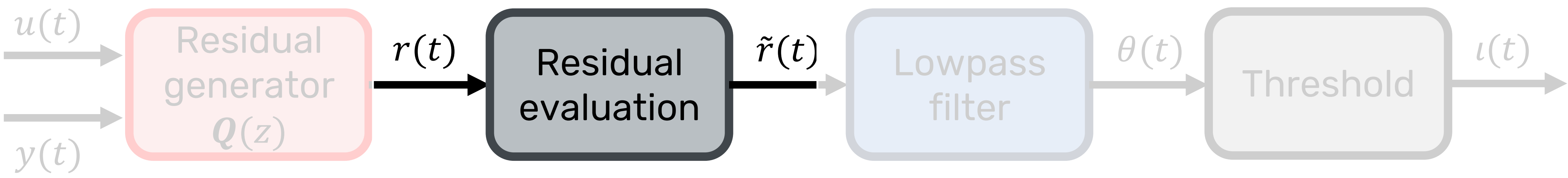


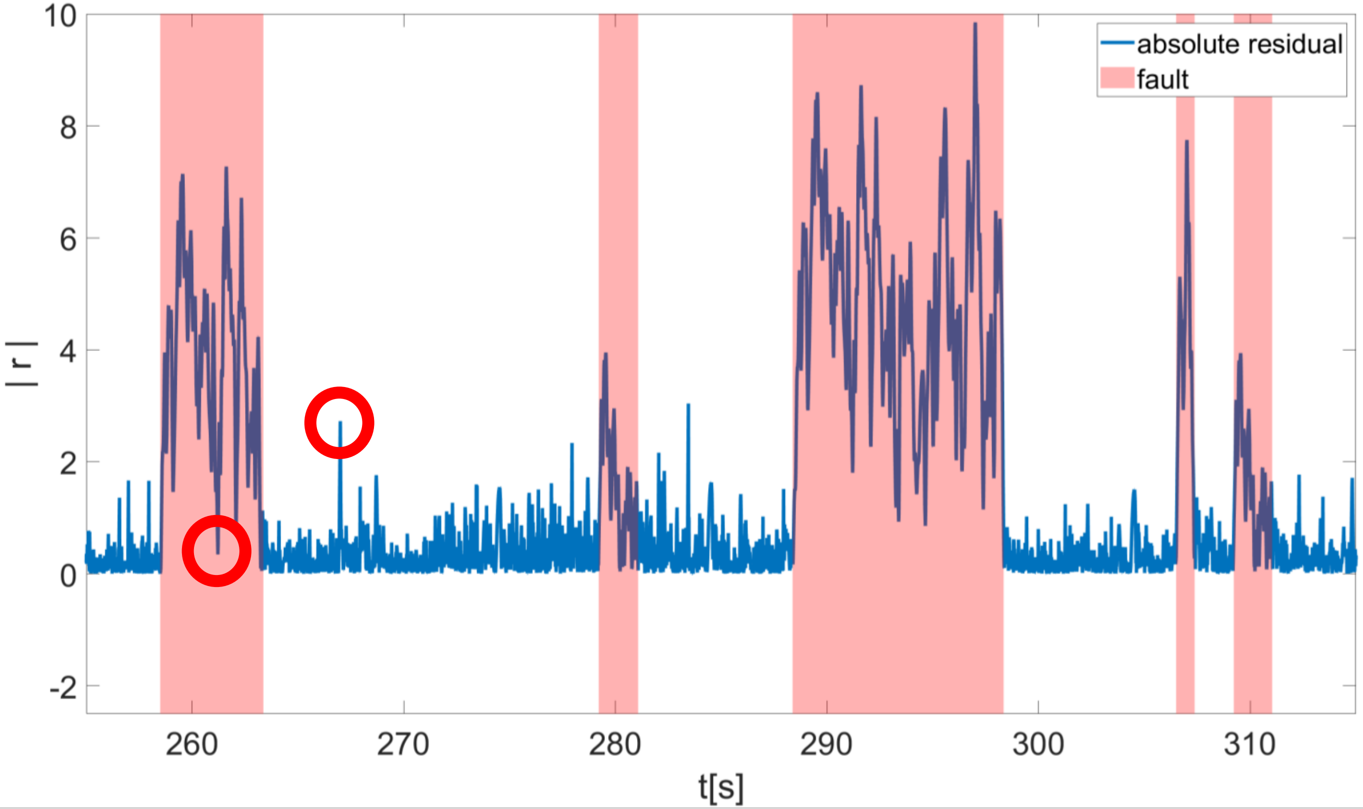
Residual generator



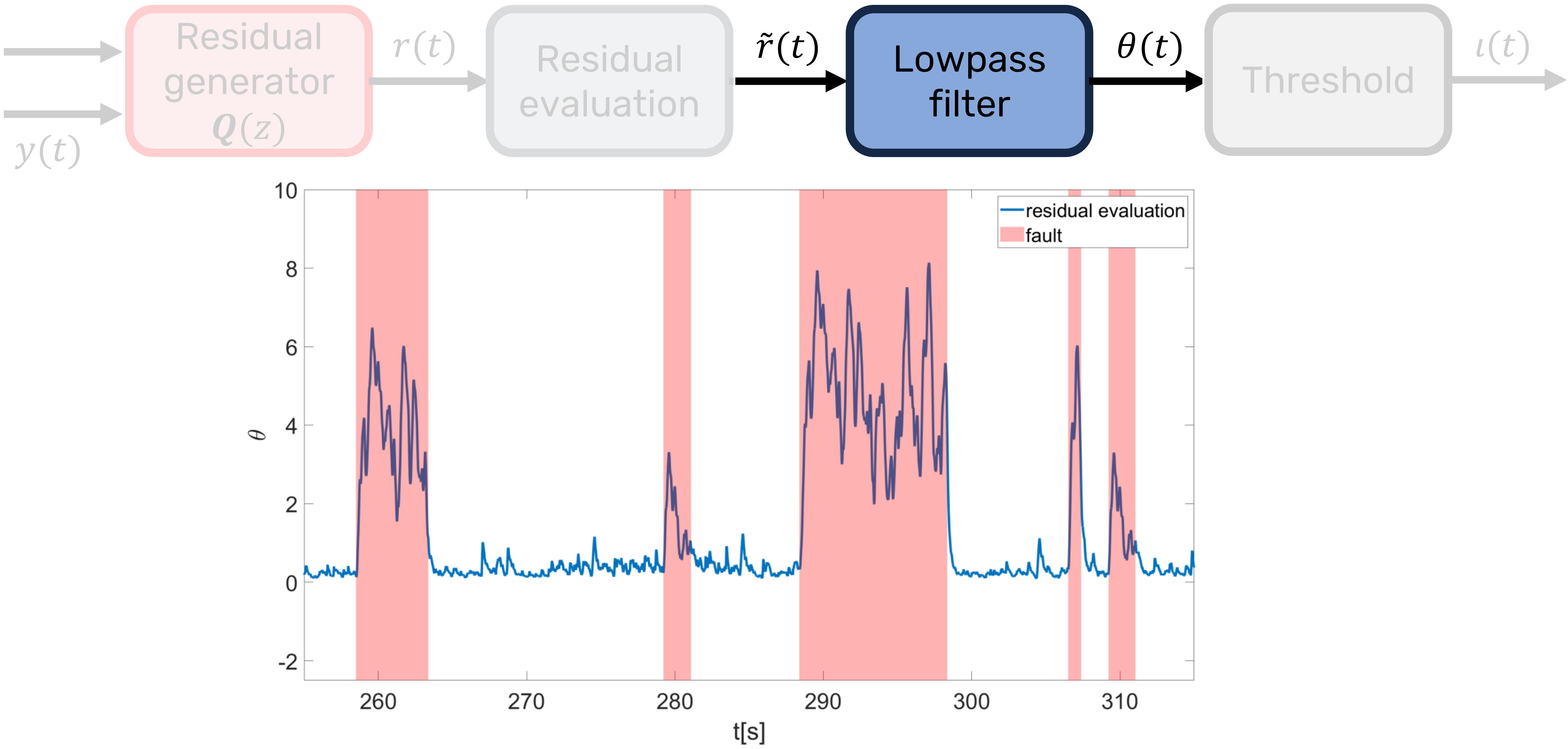


Residual generator

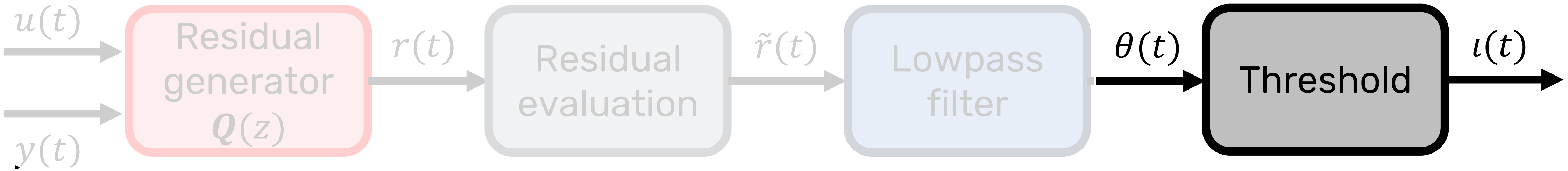


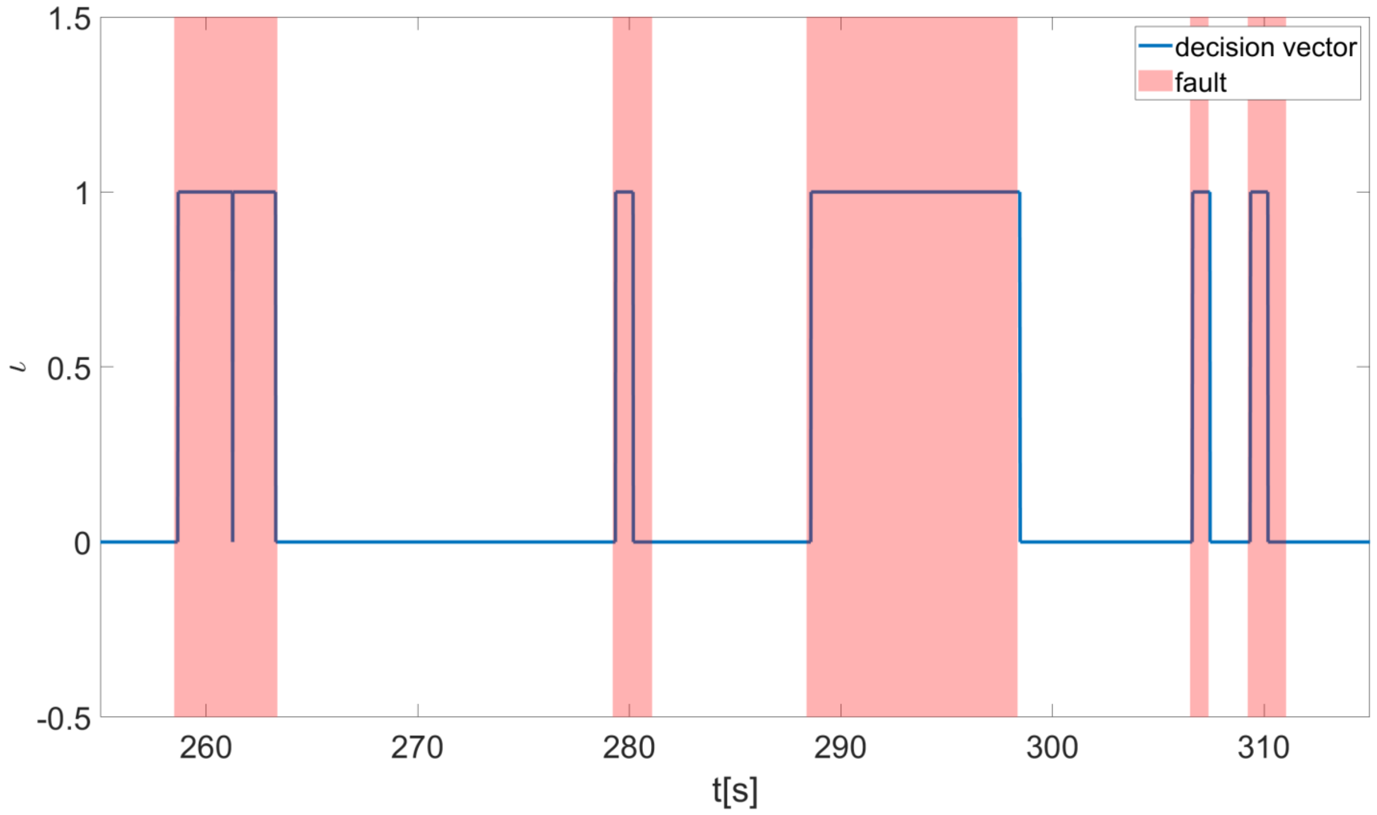


Residual generator

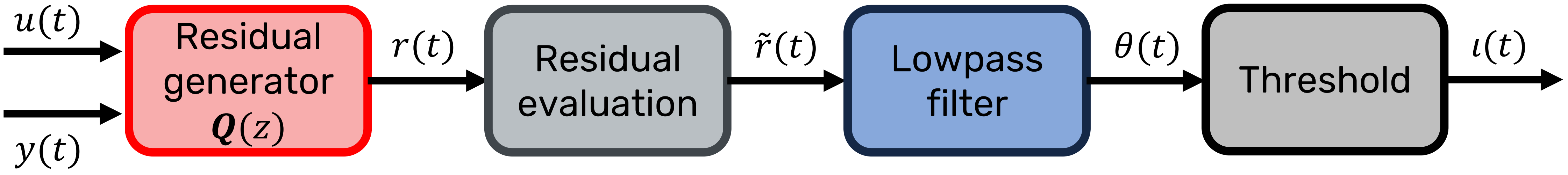


Residual generator





Residual generator



$\boldsymbol{Q}(z)$

$\tilde{r}(t) = |r(t)|$

$$H(z) = \frac{0.1139}{z - 0.8861}$$

$$\omega_t = 9.38 \text{ rad/s}$$

$\tau = 1.55$

The cut frequency of the filter and the length of the parity vector were **optimized** by minimizing a cost function, over a certain time, of the type:

$$J(\omega_t, s) = \text{false\_alarms}(\omega_t, s) \cdot \alpha + \text{missed\_alarms}(\omega_t, s) \cdot \beta$$

References

1. Ding, Steven X. “Model-based fault diagnosis techniques: design schemes, algorithms, and tools”. Springer Science & Business Media, 2008.
2. Andreas Varga “Solving Fault Diagnosis Problems - Linear Synthesis Techniques”, vol. 84 of Studies in Systems, Decision and Control, Springer International Publishing, 2017, xxviii+394 pages, hardcover: ISBN 978-3-319-51558-8, ebook: 978-3-319-51559-5.