



**UNIVERSITÀ
DEGLI STUDI
DI BERGAMO**

Dipartimento
di Ingegneria Gestionale,
dell'Informazione e della Produzione

Tutoring 1.

Linear Algebra Summary

**CONTROL AND MODELING OF
BIOLOGICAL SYSTEMS**

**MASTER DEGREE IN
MEDICAL ENGINEERING**


TEACHERS

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PLACE

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Outline

1. Vectors

2. Matrices

a. Rectangular matrices

b. Square matrices



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Introduction to vectors

A **vector** is an array of real or complex numbers.

We can distinguish two types of vectors based on the “layout” of the elements that they contain:

Row vector

$$\mathbf{x} = [x_1 \quad x_2 \quad \dots \quad x_n]$$

Column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Notation-wise, we denote a vector using a bold lower-case letter, such as \mathbf{x} , and its i -th element as x_i (not bold, still lower-case letter)

In general, if not explicitly stated otherwise, we consider \mathbf{x} to be a column vector



Introduction to vectors

Row vector

$$\mathbf{x} = [x_1 \quad x_2 \quad \dots \quad x_n]$$

Column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

In both cases we have $\mathbf{x} \in \mathbb{R}^n$ (or $\mathbf{x} \in \mathbb{C}^n$) and thus $x_i \in \mathbb{R}$ (or $x_i \in \mathbb{C}$).

$n \geq 1$ is called the **dimension** of the vector and it's the number of elements it contains. We say that \mathbf{x} is an n -dimensional vector.

We define the **transpose operator** as $(\cdot)^T$:

$$\mathbf{x} = [x_1 \quad x_2 \quad \dots \quad x_n] \longrightarrow \mathbf{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \longrightarrow \mathbf{y}^T = [y_1 \quad y_2 \quad \dots \quad y_n]$$

Vector operations

Given two n -dimensional vectors, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and two scalars $\alpha, \beta \in \mathbb{R}$, we define the following operations:

- **Sum of two vectors**

$$\mathbf{z} = \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$\mathbf{z} = \mathbf{x} + \mathbf{y} = [x_1 \quad x_2 \quad \dots \quad x_n] + [y_1 \quad y_2 \quad \dots \quad y_n] = [x_1 + y_1 \quad x_2 + y_2 \quad \dots \quad x_n + y_n]$$

In both cases we have $z_i = x_i + y_i, \forall i, 1 \leq i \leq n$ and $\mathbf{z} \in \mathbb{R}^n$

Vector operations

- **Multiplication of a vector by a scalar**

$$\mathbf{z} = \alpha \cdot \mathbf{x} = \alpha \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha \cdot x_1 \\ \alpha \cdot x_2 \\ \vdots \\ \alpha \cdot x_n \end{bmatrix}$$

$$\mathbf{z} = \alpha \cdot \mathbf{x} = \alpha \cdot [x_1 \quad x_2 \quad \dots \quad x_n] = [\alpha \cdot x_1 \quad \alpha \cdot x_2 \quad \dots \quad \alpha \cdot x_n]$$

In both cases we have $z_i = \alpha \cdot x_i, \forall i, 1 \leq i \leq n$ and $\mathbf{z} \in \mathbb{R}^n$

Vector operations

- **Linear combination of vectors:** given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define a linear combination of these two as

$$\mathbf{z} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{y}$$

This can be generalized to an arbitrary number k of vectors:

$$\mathbf{z} = \sum_{j=1}^k \alpha_j \cdot \mathbf{x}_j$$

- ✓ $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbb{R}^n$ vectors
- ✓ $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ scalars

The result is an n -dimensional vector, i.e. $\mathbf{z} \in \mathbb{R}^n$

Vector operations

- **Inner product of two vectors:** given a row vector $x \in \mathbb{R}^n$ and a column vector $y \in \mathbb{R}^n$, their inner product is defined as

$$\begin{aligned}\gamma = x \cdot y &= [x_1 \quad x_2 \quad \dots \quad x_n] \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n \cdot y_n \\ &= \sum_{i=1}^n x_i \cdot y_i\end{aligned}$$

- ✓ The result of the inner product is a scalar, $\gamma \in \mathbb{R}$
- ✓ If $\gamma = 0$, then x and y are said to be **orthogonal**
- ✓ It holds that $x \cdot y = y^T \cdot x^T$ (however $x \cdot y \neq y \cdot x$, this will be clearer shortly)



Vector operations

- **2-norm of a vector:** the 2-norm of a vector $\mathbf{x} \in \mathbb{R}^n$, denoted as $\|\mathbf{x}\|_2$ (or simply $\|\mathbf{x}\|$), is the square root of the inner product of the vector with itself

$$\|\mathbf{x}\|_2 = \sqrt{\underset{1 \times n}{\mathbf{x}^\top} \cdot \underset{n \times 1}{\mathbf{x}}} \quad \text{if } \mathbf{x} \text{ is a column vector}$$

$$\|\mathbf{x}\|_2 = \sqrt{\underset{1 \times n}{\mathbf{x}} \cdot \underset{n \times 1}{\mathbf{x}^\top}} \quad \text{if } \mathbf{x} \text{ is a row vector}$$

It is easy to see that $\|\mathbf{x}\|_2$ is a scalar since the inner product itself is a scalar

This concept can be generalized, the so-called p -norm of a vector is defined as:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Linear independence

Given k vectors $\mathbf{x}_j \in \mathbb{R}^n$, $1 \leq j \leq k$, we say that these vectors are **linearly independent** if their linear combination

$$\mathbf{z} = \alpha_1 \cdot \mathbf{x}_1 + \alpha_2 \cdot \mathbf{x}_2 + \cdots + \alpha_k \cdot \mathbf{x}_k = \sum_{j=1}^k \alpha_j \cdot \mathbf{x}_j$$

is equal to the zero-vector $\mathbf{0} = [0 \ 0 \ \dots \ 0]^\top$, i.e. $\sum_{j=1}^k \alpha_j \cdot \mathbf{x}_j = \mathbf{0}$, if and only if

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Otherwise, the vectors are said to be **linearly dependent**. If this is the case, as the name suggests, at least one of them can be written as a linear combination of the others

Linear independence: example (1)

Check if the following vectors are linearly independent

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

We need to check if the following equality holds only for $\alpha_1 = \alpha_2 = 0$

$$\alpha_1 \cdot \mathbf{x}_1 + \alpha_2 \cdot \mathbf{x}_2 = \mathbf{0}$$
$$\alpha_1 \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{cases} 2 \cdot \alpha_1 - \alpha_2 = 0 \\ 3 \cdot \alpha_1 + 2 \cdot \alpha_2 = 0 \\ -\alpha_1 + 2 \cdot \alpha_2 = 0 \end{cases}$$

Linear independence: example (1)

$$\begin{cases} 2 \cdot \alpha_1 - \alpha_2 = 0 \\ 3 \cdot \alpha_1 + 2 \cdot \alpha_2 = 0 \\ -\alpha_1 + 2 \cdot \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} \alpha_2 = 2 \cdot \alpha_1 \\ 3 \cdot \alpha_1 + 2 \cdot 2 \cdot \alpha_1 = 0 \\ -\alpha_1 + 2 \cdot 2 \cdot \alpha_1 = 0 \end{cases} \rightarrow \begin{cases} \alpha_2 = 2 \cdot \alpha_1 \\ 7 \cdot \alpha_1 = 0 \\ 3 \cdot \alpha_1 = 0 \end{cases} \\ \downarrow \\ \alpha_1 = \alpha_2 = 0$$

We can conclude that x_1 and x_2 are linearly independent

Linear independence: example (2)

Check if the following vectors are linearly independent

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$$

We need to check if the following equality holds only for $\alpha_1 = \alpha_2 = 0$

$$\alpha_1 \cdot \mathbf{x}_1 + \alpha_2 \cdot \mathbf{x}_2 = \mathbf{0}$$
$$\alpha_1 \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \longrightarrow \quad \begin{cases} 2 \cdot \alpha_1 + 4 \cdot \alpha_2 = 0 \\ 3 \cdot \alpha_1 + 6 \cdot \alpha_2 = 0 \\ -\alpha_1 - 2 \cdot \alpha_2 = 0 \end{cases}$$

Linear independence: example (2)

$$\begin{cases} 2 \cdot \alpha_1 + 4 \cdot \alpha_2 = 0 \\ 3 \cdot \alpha_1 + 6 \cdot \alpha_2 = 0 \\ -\alpha_1 - 2 \cdot \alpha_2 = 0 \end{cases}$$

This is an indeterminate system because all the equations amount to

$$\alpha_1 + 2 \cdot \alpha_2 = 0 \quad \Rightarrow \quad \text{Infinite solutions} \quad \alpha_1 = -2 \cdot \alpha_2$$

Since there exist $\alpha_1, \alpha_2 \neq 0$ for which the equality $\alpha_1 \cdot \mathbf{x}_1 + \alpha_2 \cdot \mathbf{x}_2 = \mathbf{0}$ holds, \mathbf{x}_1 and \mathbf{x}_2 are linearly dependent. As a further proof, we can see that

$$\mathbf{x}_2 = 2 \cdot \mathbf{x}_1$$

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$$



$$\begin{aligned} \alpha_1 \cdot \mathbf{x}_1 + \alpha_2 \cdot \mathbf{x}_2 &= \mathbf{0} \\ \alpha_1 \cdot \mathbf{x}_1 + \alpha_2 \cdot 2 \cdot \mathbf{x}_1 &= \mathbf{0} \\ (\alpha_1 + 2 \cdot \alpha_2) \cdot \mathbf{x}_1 &= \mathbf{0} \\ \alpha_1 &= -2 \cdot \alpha_2 \end{aligned}$$



Outline

1. Vectors

2. Matrices

a. Rectangular matrices

b. Square matrices



Introduction to matrices

Differently from a vector, which was an array with just one column or one row, a **matrix** is a collection of $m \geq 1$ **rows** and $n \geq 1$ **columns** (a sort of “table”) of real or complex numbers. A generic m -by- n -dimensional matrix A (or shortly $m \times n$ matrix) is

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Row 1

Column 2

$$A \in \mathbb{R}^{m \times n} \text{ (or } A \in \mathbb{C}^{m \times n}) \text{ and}$$
$$\text{thus } a_{ij} \in \mathbb{R} \text{ (or } a_{ij} \in \mathbb{C})$$

Notation-wise, we denote a matrix using an upper-case letter (not bold), such as A , and its (i, j) -th entry (i.e. the element on the i -th row and j -th column) as a_{ij} (lower-case letter)

Introduction to matrices

A matrix $A \in \mathbb{R}^{m \times n}$ for which the number of rows is different from the number of columns ($m \neq n$) is called a **rectangular matrix**. Vice versa, if $m = n$, then we are dealing with a **square matrix** ($A \in \mathbb{R}^{n \times n}$)

A vector is the simplest rectangular matrix

$$\mathbf{x} = [x_1 \quad x_2 \quad \dots \quad x_n] \in \mathbb{R}^{1 \times n}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Thus, many of the concepts that we will see for rectangular matrices also apply to vectors



(Rectangular) matrix operations

The **transpose operator** $(\cdot)^T$ that we have seen for vectors can be easily extended to matrices:

$$A = \begin{matrix} m \times n \\ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{matrix} \quad \longrightarrow \quad A^T = \begin{matrix} n \times m \\ \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nm} \end{bmatrix} \end{matrix}$$

$$A \in \mathbb{R}^{m \times n} \quad \xrightarrow{\text{becomes}} \quad A^T \in \mathbb{R}^{n \times m}$$
$$a_{ij} \quad \xrightarrow{\quad} \quad a_{ji}$$

Some **properties** of $(\cdot)^T$:

- ✓ $(A^T)^T = A$
- ✓ $(A + B)^T = A^T + B^T$
- ✓ $(A \cdot B)^T = B^T \cdot A^T$

Matrix transpose: example

Calculate the transpose of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

The first column of A becomes the first row of A^T and so on...

(Rectangular) matrix operations

- **Sum of two matrices:** given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, their sum $A + B$ is defined if and only if both dimensions match, i.e. $m = p$ and $n = q$. We have

$$C = A + B$$

$m \times n$ $m \times n$ $m \times n$

must match!

The entries of the resulting matrix $C \in \mathbb{R}^{m \times n}$ are calculated as:

$$c_{ij} = a_{ij} + b_{ij} \quad \begin{array}{l} \forall i, 1 \leq i \leq n \\ \forall j, 1 \leq j \leq m \end{array}$$

(Rectangular) matrix operations

- **Product of a matrix by a scalar:** given a matrix $A \in \mathbb{R}^{m \times n}$ and a scalar $\beta \in \mathbb{R}$, their product

$$C = \beta \cdot A$$

$m \times n$ $m \times n$

results in a matrix $C \in \mathbb{R}^{m \times n}$ whose entries are calculated as:

$$c_{ij} = \beta \cdot a_{ij}$$

$$\forall i, 1 \leq i \leq n$$

$$\forall j, 1 \leq j \leq m$$

(Rectangular) matrix operations

- **Matrix product:** given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, their matrix product $A \cdot B$ is defined if and only if $p = n$. We have

$$C = A \cdot B$$

$m \times q$ $m \times n$ $n \times q$

must match!

The entries of the resulting matrix $C \in \mathbb{R}^{m \times q}$ are calculated as:

$$c_{ik} = \sum_{j=1}^n a_{ij} \cdot b_{jk} \quad \begin{array}{l} \forall i, 1 \leq i \leq m \\ \forall k, 1 \leq k \leq q \end{array}$$

Remark: given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$ such that $m \neq n$, it is important to note that

$$\underbrace{A \cdot B}_{m \times m} \neq \underbrace{B \cdot A}_{n \times n}$$

Matrix product: example

Given the following matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ -7 & 8 \\ 0 & 1 \end{bmatrix}$$

Calculate $C = A \cdot B$ and $D = B \cdot A$

Firstly, we need to check the matrices' dimensions. We can see that $A \in \mathbb{R}^{2 \times 3}$ and $B \in \mathbb{R}^{3 \times 2}$ and thus we can compute both products

In particular, we will obtain $C \in \mathbb{R}^{2 \times 2}$ and $D \in \mathbb{R}^{3 \times 3}$

Matrix product: example

... and so on. We obtain:

$$C = A \cdot B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \cdot \begin{bmatrix} 1 & 3 \\ -7 & 8 \\ 0 & 1 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} -13 & 22 \\ -31 & 58 \end{bmatrix}_{2 \times 2}$$

Using the same reasoning we can also calculate:

$$D = B \cdot A = \begin{bmatrix} 1 & 3 \\ -7 & 8 \\ 0 & 1 \end{bmatrix}_{3 \times 2} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 13 & 17 & 21 \\ 25 & 26 & 27 \\ 4 & 5 & 6 \end{bmatrix}_{3 \times 3}$$

We can clearly see that $C \neq D$

Rank of a (rectangular) matrix

Given a matrix $A \in \mathbb{R}^{m \times n}$, its **rank**, denoted by $\text{rank}(A)$, is defined as the number of linearly independent columns (or equivalently rows) of A

Some important **properties** are:

- ✓ $\text{rank}(A) = \text{rank}(A^T)$
- ✓ $0 \leq \text{rank}(A) \leq \min(m, n)$. In particular, if $\text{rank}(A) = \min(m, n)$, then A is said to have **full rank**
- ✓ $\text{rank}(A) = 0$ if and only if A is a zero-matrix (which is a matrix whose entries are all zeros, $a_{ij} = 0 \forall i, j$), otherwise $\text{rank}(A) \geq 1$

Matrix rank: example

Calculate the rank of the following matrices

$$\begin{array}{c} \text{Column } \mathbf{a}_1 \\ \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 5 & 10 \end{bmatrix} \\ \text{Column } \mathbf{a}_2 \\ 3 \times 2 \end{array}$$

If A were to be full rank, we would have

$$\text{rank}(A) = \min(3, 2) = 2$$

However, we can clearly see that

$$\text{Column } \mathbf{a}_1 = 2 \cdot \text{Column } \mathbf{a}_2$$

That is, they are linearly dependent. Therefore:

$$\text{rank}(A) < 2$$

Since A is not a zero-matrix ($\text{rank}(A) \geq 1$) we can conclude that $\text{rank}(A) = 1$

$$\begin{array}{c} \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} \\ 2 \times 2 \end{array}$$

We can clearly see that

$$\alpha_1 \cdot [1 \ 5] + \alpha_2 \cdot [3 \ 7] = [0 \ 0]$$

if and only if $\alpha_1 = \alpha_2 = 0$, which means that the two rows are linearly independent.

Therefore, B has full rank:

$$\text{rank}(B) = 2$$

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Square matrices

As previously seen, square matrices are a particular type of matrices which have the same number of rows and columns

All the properties and operations that we have seen thus far for rectangular matrices also apply to square matrices. However, matrices of this type exhibit some additional and useful properties which we will now cover

Let us consider a generic square matrix $A \in \mathbb{R}^{n \times n}$, we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Antidiagonal of A ($a_{n1}, a_{(n-1)2}, \dots, a_{1n}$)

Main diagonal of A ($a_{11}, a_{22}, \dots, a_{nn}$)

Determinant of a square matrix

The **determinant** of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted by $\det(A)$, is a scalar which can be computed recursively using the so-called **Laplace expansion**. **There are two equivalent formulations:**

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

- j (column index) can be chosen freely, the summation is over i (row index)
- A_{ij} is the $(n-1) \times (n-1)$ matrix that is formed by deleting the i -th row and j -th column of A

Remark: the determinant of a scalar is the scalar itself

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \dots \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

A_{11} A_{n1}

$$\det(A) = (-1)^{1+1} \cdot a_{11} \cdot \det(A_{11}) + \dots + (-1)^{n+1} \cdot a_{n1} \cdot \det(A_{n1})$$

Laplace expansion along the first column ($j = 1$)

Determinant of a square matrix

The **determinant** of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted by $\det(A)$, is a scalar which can be computed recursively using the so-called **Laplace expansion**. **There are two equivalent formulations:**

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

- i (row index) can be chosen freely, the summation is over j (column index)
- A_{ij} is the $(n - 1) \times (n - 1)$ matrix that is formed by deleting the i -th row and j -th column of A

Remark: the determinant of a scalar is the scalar itself

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \dots \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

A_{11} A_{1n}

$$\det(A) = (-1)^{1+1} \cdot a_{11} \cdot \det(A_{11}) + \dots + (-1)^{1+n} \cdot a_{1n} \cdot \det(A_{1n})$$

Laplace expansion along the first row ($i = 1$)

Determinant of a 2×2 matrix

Let us consider a generic 2×2 matrix $A \in \mathbb{R}^{2 \times 2}$ and calculate its determinant

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = \sum_{j=1}^2 (-1)^{1+j} \cdot a_{1j} \cdot \det(A_{1j}) = (-1)^{1+1} \cdot a_{11} \cdot \det(A_{11}) + (-1)^{1+2} \cdot a_{12} \cdot \det(A_{12})$$

Laplace expansion along the first row ($i = 1$)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A_{11}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A_{12}

$$= a_{11} \cdot \det(a_{22}) - a_{12} \cdot \det(a_{21})$$

$$\det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

Determinant of a square matrix: example

Calculate the determinant of the following 3×3 matrix

$$A = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 6 & 2 \\ 1 & 7 & 3 \end{bmatrix}$$

Tip: use the Laplace expansion along the row (or column) with the most zeros. In this case, it is convenient to expand along the first column ($j = 1$)

$$\begin{aligned} \det(A) &= \sum_{i=1}^n (-1)^{i+1} \cdot a_{i1} \cdot \det(A_{i1}) \\ &= \cancel{(-1)^{1+1} \cdot a_{11} \cdot \det(A_{11})} + \\ &\quad \cancel{(-1)^{2+1} \cdot a_{21} \cdot \det(A_{21})} + \\ &\quad + (-1)^{3+1} \cdot a_{31} \cdot \det(A_{31}) \end{aligned}$$

$$A = \begin{bmatrix} \textcircled{0} & 5 & 1 \\ 0 & \boxed{6} & \boxed{2} \\ 1 & 7 & 3 \end{bmatrix}$$

A_{11}

$$A_{11} = \begin{bmatrix} 6 & 2 \\ 7 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \boxed{5} & \boxed{1} \\ \textcircled{0} & 6 & 2 \\ 1 & \boxed{7} & \boxed{3} \end{bmatrix}$$

A_{21}

$$A_{21} = \begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \boxed{5} & \boxed{1} \\ 0 & \boxed{6} & \boxed{2} \\ \textcircled{1} & 7 & 3 \end{bmatrix}$$

A_{31}

$$A_{31} = \begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}$$

Determinant of a square matrix: example

Calculate the determinant of the following 3×3 matrix

$$A = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 6 & 2 \\ 1 & 7 & 3 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \sum_{i=1}^n (-1)^{i+1} \cdot a_{i1} \cdot \det(A_{i1}) \\ &= (-1)^{3+1} \cdot a_{31} \cdot \det(A_{31}) \\ &= 1 \cdot \det \left(\begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix} \right) \\ &= 5 \cdot 2 - 6 \cdot 1 \\ &= 4 \end{aligned}$$

Properties of the determinant of a square matrix

Some important **properties of the determinant** are:

1. $\det(A^T) = \det(A), \forall A \in \mathbb{R}^{n \times n}$
2. $\det(\alpha \cdot A) = \alpha^n \cdot \det(A), \forall A \in \mathbb{R}^{n \times n}, \forall \alpha \in \mathbb{R}$
3. $\det(A \cdot B) = \det(A) \cdot \det(B), \forall A, B \in \mathbb{R}^{n \times n}$
4. $\det(A) \neq 0$ if and only if $A \in \mathbb{R}^{n \times n}$ has full rank. If $\det(A) = 0$, A is said to be a **singular matrix** (vice versa, if $\det(A) \neq 0$, A is said to be **nonsingular** or **invertible**)
5. $\det(A^{-1}) = \frac{1}{\det(A)}$ for all invertible matrices $A \in \mathbb{R}^{n \times n}$

Inverse of a square matrix

Before defining the inverse of square matrix, it is important to introduce the **identity matrix** $I_n \in \mathbb{R}^{n \times n}$ (or simply I):

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I is a square matrix whose elements on the main diagonal are ones while all the other entries are zeros

Inverse of a square matrix

Given an invertible (or nonsingular) matrix $A \in \mathbb{R}^{n \times n}$, or, in other words, a matrix A such that $\det(A) \neq 0$, there exists a unique matrix $A^{-1} \in \mathbb{R}^{n \times n}$, called the **inverse of A** , such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Let us define $B = A^{-1}$, the entries b_{ij} of the inverse of A are calculated as follows

$$b_{ij} = \frac{1}{\det(A)} \cdot (-1)^{j+i} \cdot \det(A_{ji})$$

Be careful: the order is flipped!

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} (-1)^{1+1} \cdot \det(A_{11}) & \cdots & (-1)^{n+1} \cdot \det(A_{n1}) \\ \vdots & \ddots & \vdots \\ (-1)^{1+n} \cdot \det(A_{1n}) & \cdots & (-1)^{n+n} \cdot \det(A_{nn}) \end{bmatrix}$$

Remark: given two invertible matrices $A, B \in \mathbb{R}^{n \times n}$ it holds that $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

Inverse of a 2×2 matrix

Let us consider a generic 2×2 matrix $A \in \mathbb{R}^{2 \times 2}$ and calculate its inverse

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} (-1)^{1+1} \cdot \det(A_{11}) & (-1)^{2+1} \cdot \det(A_{21}) \\ (-1)^{1+2} \cdot \det(A_{12}) & (-1)^{2+2} \cdot \det(A_{22}) \end{bmatrix}$$

$$A^{-1} = \frac{1}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

An easy way to remember this formula is to “switch the entries on the main diagonal of A and flip the sign of those on the antidiagonal”

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A_{11}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A_{12} A_{21}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A_{22}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A_{21}

Inverse of a square matrix: example

Calculate the inverse of the following 3×3 matrix

$$A = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 6 & 2 \\ 1 & 7 & 3 \end{bmatrix}$$

In the previous example, we have seen that $\det(A) = 4 \neq 0$ and thus A is invertible

The inverse of A is computed as

$$A^{-1} = \frac{1}{4} \cdot \begin{bmatrix} (-1)^{1+1} \cdot \det(A_{11}) & (-1)^{2+1} \cdot \det(A_{21}) & (-1)^{3+1} \cdot \det(A_{31}) \\ (-1)^{1+2} \cdot \det(A_{12}) & (-1)^{2+2} \cdot \det(A_{22}) & (-1)^{3+2} \cdot \det(A_{32}) \\ (-1)^{1+3} \cdot \det(A_{13}) & (-1)^{2+3} \cdot \det(A_{23}) & (-1)^{3+3} \cdot \det(A_{33}) \end{bmatrix}$$

Inverse of a square matrix: example

$$\det(A_{11}) = \det\left(\begin{bmatrix} 6 & 2 \\ 7 & 3 \end{bmatrix}\right) = 4$$

$$\det(A_{12}) = \det\left(\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}\right) = -2$$

$$\det(A_{13}) = \det\left(\begin{bmatrix} 0 & 6 \\ 1 & 7 \end{bmatrix}\right) = -6$$

$$\det(A_{21}) = \det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right) = 8$$

$$\det(A_{22}) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\right) = -1$$

$$\det(A_{23}) = \det\left(\begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}\right) = -5$$

$$\det(A_{31}) = \det\left(\begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}\right) = 4$$

$$\det(A_{32}) = \det\left(\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right) = 0$$

$$\det(A_{33}) = \det\left(\begin{bmatrix} 0 & 5 \\ 0 & 6 \end{bmatrix}\right) = 0$$

$$A = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 6 & 2 \\ 1 & 7 & 3 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{4} \cdot \begin{bmatrix} (-1)^{1+1} \cdot \det(A_{11}) & (-1)^{2+1} \cdot \det(A_{21}) & (-1)^{3+1} \cdot \det(A_{31}) \\ (-1)^{1+2} \cdot \det(A_{12}) & (-1)^{2+2} \cdot \det(A_{22}) & (-1)^{3+2} \cdot \det(A_{32}) \\ (-1)^{1+3} \cdot \det(A_{13}) & (-1)^{2+3} \cdot \det(A_{23}) & (-1)^{3+3} \cdot \det(A_{33}) \end{bmatrix} \\ &= \frac{1}{4} \cdot \begin{bmatrix} 4 & -(8) & 4 \\ -(-2) & -1 & 0 \\ -6 & -(-5) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{3}{2} & \frac{5}{4} & 0 \end{bmatrix} \end{aligned}$$

Eigenvalues and eigenvectors of a square matrix

An **eigenvalue** of a square matrix $A \in \mathbb{R}^{n \times n}$ is a scalar $\lambda \in \mathbb{C}$ that satisfies

$$\underset{n \times n}{A} \cdot \underset{n \times 1}{\boldsymbol{v}} = \lambda \cdot \underset{n \times 1}{\boldsymbol{v}}$$

for some nonzero vector $\boldsymbol{v} \in \mathbb{C}^n$. All nonzero vectors that satisfy the above equation are called the **eigenvectors** of A corresponding to the eigenvalue λ

Some important **remarks**:

- An $n \times n$ matrix can have at most n different eigenvalues (in any case, A always has n eigenvalues but some of them might be repeated)
- Even if A is a real-valued matrix, i.e. $A \in \mathbb{R}^{n \times n}$, the eigenvalues and eigenvectors can be complex-valued
- Eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$, $k \leq n$ of a matrix $A \in \mathbb{R}^{n \times n}$ are linearly independent. Eigenvectors corresponding to repetitive eigenvalues can be linearly independent, but this is not necessary



Eigenvalues and eigenvectors of a square matrix

We define the **characteristic polynomial** of A as

$$\pi(\lambda) = \det(\lambda \cdot I - A)$$

$n \times n$ $n \times n$

The eigenvalues of A are the roots of the **characteristic equation**

$$\pi(\lambda) = 0$$

Lemma: suppose that a matrix $A \in \mathbb{R}^{n \times n}$ has a complex-valued eigenvalue λ , with corresponding eigenvector v . Then, the complex conjugate of λ is also an eigenvalue of A and the complex conjugate of v is its corresponding eigenvector

Property: given the n eigenvalues of A , $\lambda_1, \dots, \lambda_n$, it is possible to demonstrate that

$$\det(A) = \lambda_1 \cdot \dots \cdot \lambda_n = \prod_i^n \lambda_i$$

Eigenvalues and eigenvectors: example (1)

Calculate the eigenvalues and their associated eigenvectors of the following 2×2 matrix

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$$

We solve the characteristic equation

$$\det(\lambda \cdot I - A) = 0$$

$$\det\left(\lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda + 2 & -3 \\ -1 & \lambda + 4 \end{bmatrix}\right) = 0$$

$$(\lambda + 2) \cdot (\lambda + 4) - 3 = 0$$

$$\lambda^2 + 6 \cdot \lambda + 5 = 0$$

$$(\lambda + 5) \cdot (\lambda + 1) = 0$$

Thus, the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -1$



Eigenvalues and eigenvectors: example (1)

To calculate the eigenvectors associated to λ_1 and λ_2 we need to solve the following equations

$$A \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} v_a \\ v_b \end{bmatrix} = -5 \cdot \begin{bmatrix} v_a \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} -2 \cdot v_a + 3 \cdot v_b \\ v_a - 4 \cdot v_b \end{bmatrix} = -5 \cdot \begin{bmatrix} v_a \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} -2 \cdot v_a + 3 \cdot v_b + 5 \cdot v_a \\ v_a - 4 \cdot v_b + 5 \cdot v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3 \cdot v_a + 3 \cdot v_b = 0 \\ v_a + v_b = 0 \\ v_a = -v_b \\ v_a = -v_b \end{cases} \longrightarrow \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

By choosing for example $v_b = 1$

$$A \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2$$

$$\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} v_c \\ v_d \end{bmatrix} = - \begin{bmatrix} v_c \\ v_d \end{bmatrix}$$

$$\begin{bmatrix} -2 \cdot v_c + 3 \cdot v_d \\ v_c - 4 \cdot v_d \end{bmatrix} = - \begin{bmatrix} v_c \\ v_d \end{bmatrix}$$

$$\begin{bmatrix} -2 \cdot v_c + 3 \cdot v_d + v_c \\ v_c - 4 \cdot v_d + v_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -v_c + 3 \cdot v_d = 0 \\ v_c - 3 \cdot v_d = 0 \\ v_c = 3 \cdot v_d \\ v_c = 3 \cdot v_d \end{cases} \longrightarrow \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

By choosing for example $v_d = 1$

Eigenvalues and eigenvectors: example (2)

Calculate the eigenvalues and their associated eigenvectors of the following 2×2 matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We solve the characteristic equation

$$\det(\lambda \cdot I - A) = 0$$

$$\det\left(\lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 1 \end{bmatrix}\right) = 0$$

$$(\lambda - 1)^2 + 1 = 0$$

$$\lambda^2 - 2 \cdot \lambda + 2 = 0$$

$$\sqrt{\Delta} = \sqrt{(-2)^2 - 4 \cdot 2} = \sqrt{-4} = 2j$$

$$\lambda_{1,2} = \frac{2 \pm 2j}{2} = 1 \pm j$$

Thus, the eigenvalues are $\lambda_1 = 1 + j$ and its complex conjugate $\lambda_2 = 1 - j$



Eigenvalues and eigenvectors: example (2)

To calculate the eigenvectors associated to λ_1 and λ_2 we need to solve the following equations

$$A \cdot v_1 = \lambda_1 \cdot v_1$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_a \\ v_b \end{bmatrix} = (1 + j) \cdot \begin{bmatrix} v_a \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} v_a - v_b \\ v_a + v_b \end{bmatrix} = (1 + j) \cdot \begin{bmatrix} v_a \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} v_a - v_b - v_a - j \cdot v_a \\ v_a + v_b - v_b - j \cdot v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -j \cdot v_a - v_b = 0 \\ v_a - j \cdot v_b = 0 \end{cases}$$

$$\begin{cases} v_a = j \cdot v_b \\ v_a = j \cdot v_b \end{cases} \longrightarrow v_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

Remember that $\frac{1}{j} = -j$

complex conjugate of

$$A \cdot v_2 = \lambda_2 \cdot v_2$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_c \\ v_d \end{bmatrix} = (1 - j) \cdot \begin{bmatrix} v_c \\ v_d \end{bmatrix}$$

$$\begin{bmatrix} v_c - v_d \\ v_c + v_d \end{bmatrix} = (1 - j) \cdot \begin{bmatrix} v_c \\ v_d \end{bmatrix}$$

$$\begin{bmatrix} v_c - v_d - v_c + j \cdot v_c \\ v_c + v_d - v_d + j \cdot v_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} j \cdot v_c - v_d = 0 \\ v_c + j \cdot v_d = 0 \end{cases}$$

$$\begin{cases} v_c = -j \cdot v_d \\ v_c = -j \cdot v_d \end{cases} \longrightarrow v_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

By choosing for example $v_b = -j$

v_d must be chosen appropriately, $v_d = j$



Triangular and diagonal square matrices

There exist some specific types of square matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$$

Upper triangular matrix

(Some) nonzero entries on and above the main diagonal

$$A = \begin{bmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Lower triangular matrix

(Some) nonzero entries on and below the main diagonal

$$A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$$

Diagonal matrix

(Some) nonzero entries only on the main diagonal

Often denoted as:

$$A = \text{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$$

These matrices exhibit some important **properties**:

- $\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$
- The eigenvalues are the entries on the main diagonal, $\lambda_i = a_{ii}$

Diagonalization of a square matrix

Vice versa, the starting matrix A can be computed as

$$A = T_D \cdot A_D \cdot T_D^{-1}$$

Corollary: if A has n distinct eigenvalues, then A is diagonalizable

Note: the opposite of this corollary is false, i.e. if A is diagonalizable it does not imply that it has n distinct eigenvalues. For example, the identity matrix is such that $\lambda_1 = \lambda_2 = \dots = \lambda_n = 1$, yet it is diagonalizable

Diagonalizable matrices play a key-role in the computation of the matrix power and the matrix exponential which in turn are necessary tools to analyze the stability of dynamical systems



Diagonalization of a matrix: example

Diagonalize the following 2×2 matrix

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$$

In the previous example we have computed the eigenvalues and eigenvectors of A

$$\lambda_1 = -5$$

$$\lambda_2 = -1$$

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Matrix A has $n = 2$ distinct eigenvalues and is thus diagonalizable. We construct the matrix T_D :

$$T_D = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

Diagonalization of a matrix: example

We can now compute the diagonalized matrix A_D

$$\begin{aligned}A_D &= T_D^{-1} \cdot A \cdot T_D = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{-1-3} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -3 \\ -5 & -1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} 20 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

$\lambda_1 \leftarrow$ (circled -5) \rightarrow (circled -1) λ_2

Power of a square matrix

Given a square matrix $A \in \mathbb{R}^{n \times n}$ and a scalar integer $k \geq 0$, we define the **k -th power of the matrix** A as

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_{k \text{ times}}$$

Remark: $A^0 = I$

Now suppose that A is a diagonal matrix, $A = \text{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$, the computation of its power becomes much simpler:

$$A^k = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \cdot \dots \cdot \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^k & & & \\ & a_{22}^k & & \\ & & \ddots & \\ & & & a_{nn}^k \end{bmatrix}$$

$$A^k = \text{diag}\{a_{11}^k, a_{22}^k, \dots, a_{nn}^k\}$$

Power of a square matrix

Instead of imposing A diagonal, we could only suppose A to be diagonalizable. That is, A can be factorized as

$$A = T_D \cdot A_D \cdot T_D^{-1}$$

$$A_D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Let us substitute this expression inside A^k

$$A^k = A \cdot A \cdot \dots \cdot A = (T_D \cdot A_D \cdot T_D^{-1}) \cdot (T_D \cdot A_D \cdot T_D^{-1}) \cdot \dots \cdot (T_D \cdot A_D \cdot T_D^{-1})$$

$$T_D^{-1} \cdot T_D = I$$

$$= T_D \cdot A_D \cdot A_D \cdot \dots \cdot A_D \cdot T_D^{-1}$$

$$= T_D \cdot A_D^k \cdot T_D^{-1}$$

$$A^k = T_D \cdot \text{diag}\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\} \cdot T_D^{-1}$$

**Useful to demonstrate
the stability of discrete
time dynamical systems**

Power of a matrix: example

Calculate the 3rd power of the following matrix

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$$

In the previous example we have diagonalized this matrix and obtained

$$A_D = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \quad T_D = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \quad T_D^{-1} = -\frac{1}{4} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix}$$

We can use these matrices to compute the power

$$\begin{aligned} A^3 &= T_D \cdot A_D^3 \cdot T_D^{-1} = -\frac{1}{4} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} (-5)^3 & 0 \\ 0 & (-1)^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -125 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -125 & 375 \\ 1 & 1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} 128 & -372 \\ -124 & 376 \end{bmatrix} = \begin{bmatrix} -32 & 93 \\ 31 & -94 \end{bmatrix} \end{aligned}$$

Exponential of a square matrix

Given a square matrix $A \in \mathbb{R}^{n \times n}$, we define its **exponential** as

$$e^A = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot (A)^i = I + A + \frac{1}{2} \cdot A^2 + \dots$$

Now suppose that A is a diagonal matrix, $A = \text{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$, it is possible to demonstrate that

$$e^A = \begin{bmatrix} e^{a_{11}} & & & \\ & e^{a_{22}} & & \\ & & \ddots & \\ & & & e^{a_{nn}} \end{bmatrix}$$

$$e^A = \text{diag}\{e^{a_{11}}, e^{a_{22}}, \dots, e^{a_{nn}}\}$$

Exponential of a square matrix

Instead of imposing A diagonal, we could only suppose A to be diagonalizable. That is, A can be factorized as

$$A = T_D \cdot A_D \cdot T_D^{-1}$$

$$A_D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Let us substitute this expression inside e^A

$$\begin{aligned} e^A &= \sum_{i=0}^{\infty} \frac{1}{i!} \cdot (A)^i = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot (T_D \cdot A_D \cdot T_D^{-1})^i \\ &= I + T_D \cdot A_D \cdot T_D^{-1} + \frac{1}{2} \cdot (T_D \cdot A_D \cdot T_D^{-1}) \cdot \underbrace{(T_D^{-1} \cdot T_D)}_{= I} \cdot (T_D \cdot A_D \cdot T_D^{-1}) + \dots \\ &= I + T_D \cdot A_D \cdot T_D^{-1} + \frac{1}{2} \cdot T_D \cdot A_D^2 \cdot T_D^{-1} + \dots \\ &= T_D \cdot \left(I + A_D + \frac{1}{2} A_D^2 + \dots \right) \cdot T_D^{-1} = T_D \cdot e^{A_D} \cdot T_D^{-1} \end{aligned}$$

Useful to demonstrate the stability of continuous time dynamical systems

$$e^A = T_D \cdot \text{diag}\{e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}\} \cdot T_D^{-1}$$

exponential of a diagonal matrix!

Exponential of a matrix: example

Calculate the exponential of the following matrix

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$$

In the previous example we have diagonalized this matrix and obtained

$$A_D = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \quad T_D = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \quad T_D^{-1} = -\frac{1}{4} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix}$$

We can use these matrices to compute the exponential

$$\begin{aligned} e^A &= T_D \cdot e^{A_D} \cdot T_D^{-1} = -\frac{1}{4} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \cdot e^{\begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-5} & 0 \\ 0 & e^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-5} & -3 \cdot e^{-5} \\ -e^{-1} & -e^{-1} \end{bmatrix} \\ &= -\frac{1}{4} \cdot \begin{bmatrix} -e^{-5} - 3 \cdot e^{-1} & 3 \cdot e^{-5} - 3 \cdot e^{-1} \\ e^{-5} - e^{-1} & -3 \cdot e^{-5} - e^{-1} \end{bmatrix} = \begin{bmatrix} 0.2776 & 0.2709 \\ 0.0903 & 0.0970 \end{bmatrix} \end{aligned}$$

Other (minor) definitions for square matrices

- The **trace** of a square matrix $A \in \mathbb{R}^{n \times n}$ is the sum of its main diagonal entries

$$\text{tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i$$

- ✓ a_{ii} : i -th entry on the main diagonal of A
- ✓ λ_i : i -th eigenvalue of A

- A square matrix $A \in \mathbb{R}^{n \times n}$ that satisfies $A^T = A$ is called a **symmetric** matrix
- A square matrix $A \in \mathbb{R}^{n \times n}$ that satisfies $A^T \cdot A = A \cdot A^T = I$ is called an **orthogonal** matrix. Moreover, we have that $A^T = A^{-1}$
- A square matrix $A \in \mathbb{R}^{n \times n}$ is called **positive-definite** if $\forall x \in \mathbb{R}^n$ it satisfies $x^T \cdot A \cdot x > 0$
- A square matrix $A \in \mathbb{R}^{n \times n}$ is called **positive-semidefinite** if $\forall x \in \mathbb{R}^n$ it satisfies $x^T \cdot A \cdot x \geq 0$
- A square matrix $A \in \mathbb{R}^{n \times n}$ is called **negative-definite** if $\forall x \in \mathbb{R}^n$ it satisfies $x^T \cdot A \cdot x < 0$
- A square matrix $A \in \mathbb{R}^{n \times n}$ is called **negative-semidefinite** if $\forall x \in \mathbb{R}^n$ it satisfies $x^T \cdot A \cdot x \leq 0$



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