UNIVERSITÀ DEGLI STUDI DI BERGAMO
Dipartimento di Ingegneria Gestionale, dell'Informazione e della Produzione

Lesson 7.

Laplace transform
Transfer function
of continuous time systems

CONTROL AND MODELING OF BIOLOGICAL SYSTEMS

MASTER DEGREE IN MEDICAL ENGINEERING

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Outline

1. Laplace Transform

2. Transfer function

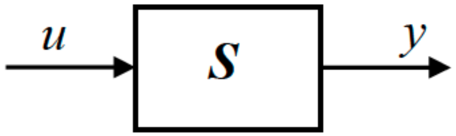
Outline

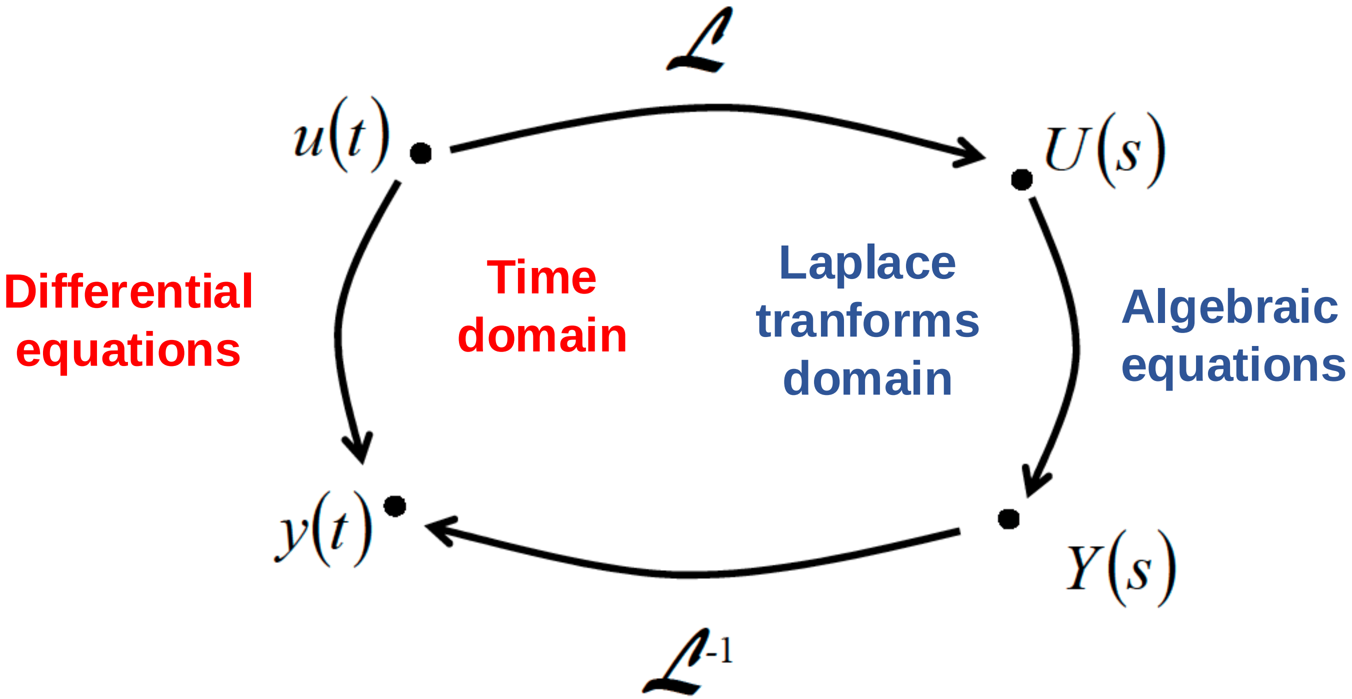
1. **Laplace Transform**

2. Transfer function

Laplace transform

Given a **continuous-time system**





Laplace transform

Given a **continuous-time signal** $f(t) : \mathbb{R} \rightarrow \mathbb{R}$, the Laplace transform is a function $F(s) : \mathbb{C} \rightarrow \mathbb{R}$

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$$

where $s \in \mathbb{C}$ is a complex number.

Why another transformation?

- It converts a continuous time signal, which is a real function, into a real function of a complex variable.

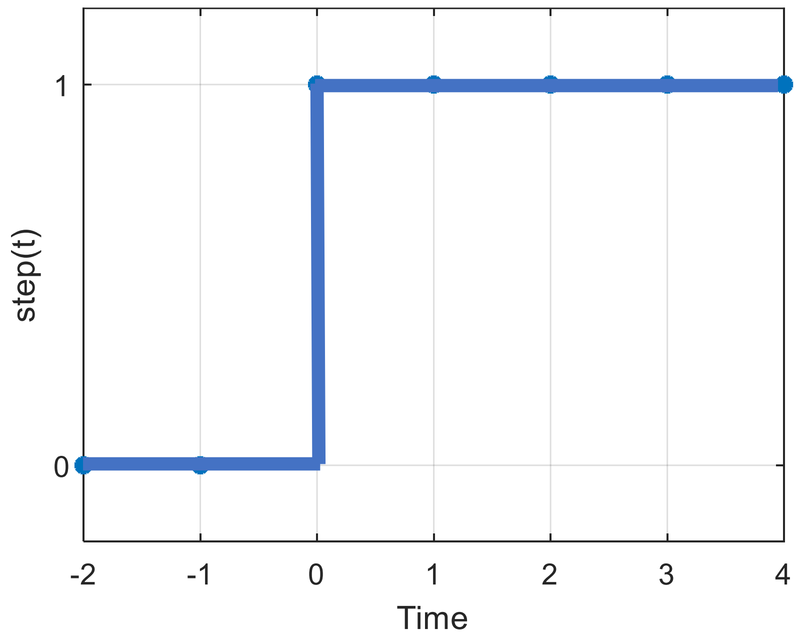
- $\mathcal{L}[f(t)]$ is not defined for the values of $f(t)$ for $t < 0$.

- Roughly speaking, it can be seen as the continuous version of the Z-transform.

- The Laplace transform has some interesting properties that allow us to analyze and solve continuous time dynamical systems.

Example: step

$$step(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}[step(t)](s) = \int_0^\infty step(t) e^{-st} dt$$

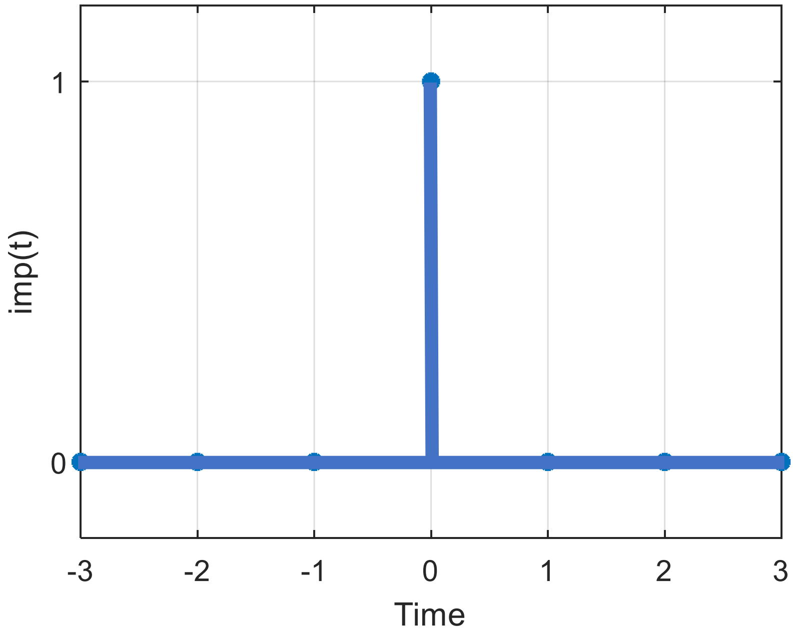
$$= \left[-\frac{e^{-st}}{s} \right]_0^\infty$$

$$= 0 + \frac{1}{s}$$

$$= \frac{1}{s}$$

Example: impulse

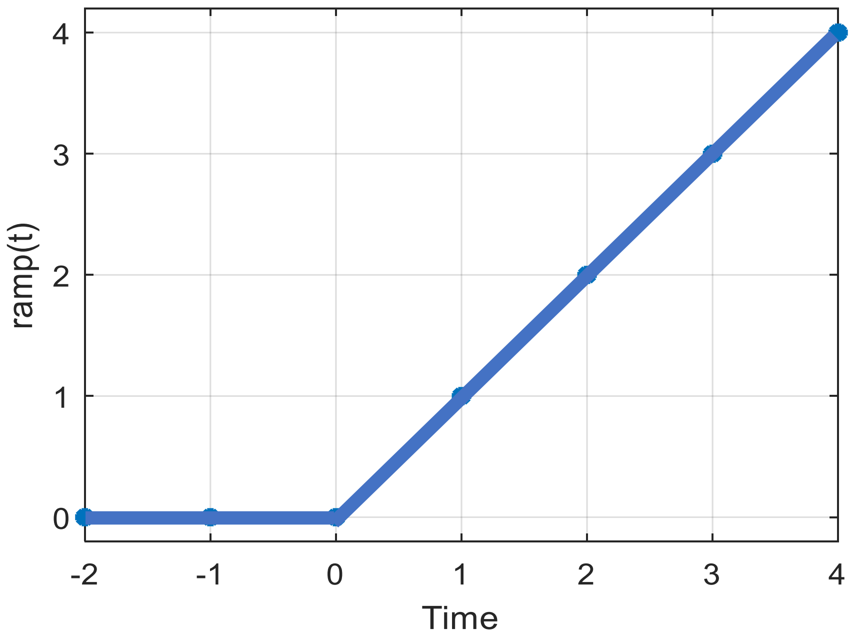
$$imp(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$



$$\mathcal{L}[imp(t)](s) = \int_0^{\infty} imp(t)e^{-st}dt = 1$$

Example: ramp

$$ramp(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}[ramp(t)](s) = \int_0^\infty ramp(t)e^{-st}dt$$

$$= \int_0^\infty te^{-st}dt$$

$$= \frac{1}{s^2}\left[-se^{-st}t - e^{-st}\right]_0^\infty$$

$$= \frac{1}{s^2}$$

Linearity

Given two signals $f_1(t)$ and $f_2(t)$ and $\alpha, \beta \in \mathbb{C}$ then the Laplace transform of the signal:

$$g(t) = \alpha \cdot f_1(t) + \beta \cdot f_2(t)$$

is given by:

$$\mathcal{L}[g(t)](s) = \int_0^\infty g(t)e^{-st}dt = \int_0^\infty [\alpha \cdot f_1(t) + \beta \cdot f_2(t)]e^{-st}dt$$

$$= \alpha \boxed{\int_0^\infty f_1(t)e^{-st}dt} + \beta \boxed{\int_0^\infty f_2(t)e^{-st}dt}$$

$$= \alpha \cdot \mathcal{L}[f_1](s) + \beta \cdot \mathcal{L}[f_2](s)$$

Delay

Given a signal $f(t)$ then the Laplace transform of the signal:

$$g(t) = f(t-\tau)$$

is given by:

$$\mathcal{L}[g(t)](s) = \int_0^{\infty} g(t)e^{-st}dt =$$

$$= \int_0^{\infty} f(t-\tau)e^{-st}dt =$$

$$= e^{-s\tau}\mathcal{L}[f(t)](s)$$

Example

$$g(t) = 2step(t) - step(t-4)$$

Then:

$$\mathcal{L}[g(t)](s) = \int_0^{\infty} g(t)e^{-st}dt =$$

$$= \int_0^{\infty} [2step(t) - step(t-4)]e^{-st}dt =$$

$$= 2\int_0^{\infty} step(t)e^{-st}dt - \int_0^{\infty} step(t-4)e^{-st}dt = \frac{2}{s} - \frac{e^{-4s}}{s}$$

Translation in the transform domain

Given a signal $f(t)$ then the Laplace transform of the signal:

$$g(t) = e^{at} f(t)$$

Is given by:

$$\mathcal{L}[g(t)](s) = \int_0^\infty g(t) e^{-st} dt =$$

$$= \int_0^\infty e^{at} f(t) e^{-st} dt = \int_0^\infty f(t) e^{(a-s)t} dt$$

$$= \mathcal{L}[f(t)](s-a)$$

Example

$$g(t) = e^{at} step(t)$$

Then:

$$\mathcal{L}[g(t)](s) = \int_0^{\infty} g(t) e^{-st} dt =$$

$$= \int_0^{\infty} [e^{at} step(t)] e^{-st} dt =$$

$$= \int_0^{\infty} step(t) e^{(a-s)t} dt = \frac{1}{s-a}$$

Transform domain derivation

Given a signal $f(t)$ and its Laplace transform $F(s)$:

$$F(s) = \mathcal{L}[f(t)](s)$$

Then:

$$-\frac{d}{ds}F(s) = -\int_0^{\infty} f(t)\frac{de^{-st}}{ds}dt = -\int_0^{\infty} -tf(t)e^{-st}dt = \mathcal{L}[tf(t)](s)$$

Time domain derivation

Given a signal $f(t)$ then the Laplace transform of the signal:

$$g(t) = \frac{d}{dt} f(t)$$

Is given by:

$$\mathcal{L}[g(t)](s) = s\mathcal{L}[f(t)] - f(0)$$

Time domain integration

Given a signal $f(t)$ then the Laplace transform of the signal:

$$g(t) = \int_0^t f(\tau)\, d\tau$$

Is given by:

$$\mathcal{L}[g(t)](s) = \frac{1}{s}\mathcal{L}[f(t)]$$

The Laplace transform converts derivation and integration into multiplications. Thus, it can be used to solve differential equations.

Initial and final value theorems

➢ Given a signal $f(t)$ and defining $F(s)$ its Laplace transform, then:

$$\lim_{s \to +\infty} sF(s) = f(0)$$

➢ Given a signal $f(t)$ and defining $F(s)$ its Laplace transform, then:

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)$$

Important transform

$$f_1(t) = imp(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_1(t)](s) = 1$$

$$f_2(t) = step(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t \neq 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_2(t)](s) = \frac{1}{s}$$

$$f_3(t) = ramp(t) = \begin{cases} t & t \geq 0 \\ 0 & t \neq 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_3(t)](s) = \frac{1}{s^2}$$

$$f_4(t) = par(t) = \begin{cases} t^2 & t \geq 0 \\ 0 & t \neq 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_4(t)](s) = \frac{1}{s^3}$$

Important transform

$$f_5(t) = ex_a(t) = \begin{cases} e^{at} & t \geq 0 \\ 0 & t \neq 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_5(t)](s) = \frac{1}{s-a}$$

$$f_6(t) = \begin{cases} \sin(\omega \cdot t) & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_6(t)](s) = \frac{\omega}{s^2+\omega^2}$$

$$f_7(t) = \begin{cases} \cos(\omega \cdot t) & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \Longleftrightarrow \quad \mathcal{L}[f_7(t)](s) = \frac{s}{s^2+\omega^2}$$

Polynomial form

Given a signal $f(t)$ then:

$$\mathcal{L}[f(t)](s) = \frac{N(s)}{D(s)}$$

Where $N(s)$ and $D(s)$ are finite degree polynomials.

- The roots of $D(s)$ are called *poles* of the signal
- The roots of $N(s)$ are called *zeros* of the signal
- Poles and zeros are *singularities* of $\mathcal{L}[f(t)](s)$.

Inverse Transform

Given a Laplace transform $F(s)$, then the original function can be derived by solving:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

For all $f(t) = 0$, when $t \leq 0$.

- The inverse transform can be easily obtained using the Heaviside decomposition **Not treated in this course**

Outline

1. Z-Transform

2. **Transfer function**

Laplace transform of the movements

Consider the state movement of the continuous-time LTI system:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A \cdot \boldsymbol{x}(t) + B \cdot u(t) \\ y(t) = C \cdot \boldsymbol{x}(t) + D \cdot u(t) \end{cases}$$

Corresponding at the input $u(t) = \check{u}(t),\ t \geq 0$ and initial state $\check{\boldsymbol{x}}(0) = 0$.

We want to compute:

- $\mathcal{L}\,[\check{\boldsymbol{x}}](s)$: the Laplace-transform of the forced state movement
- $\mathcal{L}\,[\check{y}](s)$: the Laplace-transform of the forced output movement

Forced state movement

We know that:

$$\boldsymbol{f}(t) \triangleq \dot{\breve{\boldsymbol{x}}}(t) = A \cdot \breve{\boldsymbol{x}}(t) + B \cdot \breve{u}(t)$$

Therefore:

$$\mathcal{L}\,[\boldsymbol{f}](s) = A \cdot \mathcal{L}[\breve{\boldsymbol{x}}](s) + B \cdot \mathcal{L}[\breve{u}](s)$$

$$\mathrm{s} \cdot \mathcal{L}[\breve{\boldsymbol{x}}](s) - \breve{\boldsymbol{x}}(0) = A \cdot \mathcal{L}[\breve{\boldsymbol{x}}](s) + B \cdot \mathcal{L}[\breve{u}](s)$$

$$\mathrm{s} \cdot \mathcal{L}[\breve{\boldsymbol{x}}](s) - A \cdot \mathcal{L}[\breve{\boldsymbol{x}}](s) = B \cdot \mathcal{L}[\breve{u}](s)$$

$$(\mathrm{s} \cdot I_n - A) \cdot \mathcal{L}[\breve{\boldsymbol{x}}](s) = B \cdot \mathcal{L}[\breve{u}](s)$$

$$\mathcal{L}[\breve{\boldsymbol{x}}](s) = (\mathrm{s} \cdot I_n - A)^{-1} \cdot B \cdot \mathcal{L}[\breve{u}](s)$$

Forced output movement

We know that:

$$\check{y}(t) = C \cdot \boldsymbol{x}(t) + D \cdot \check{u}(t)$$

Therefore:

$$\begin{aligned}
\mathcal{L}\,[\check{y}](s) &= C \cdot \mathcal{L}[\check{\boldsymbol{x}}](s) + D \cdot \mathcal{L}[\check{u}](s) \\
&= C \cdot (\mathrm{s} \cdot I_n - A)^{-1} \cdot B \cdot \mathcal{L}[\check{u}](s) + D \cdot \mathcal{L}[\check{u}](s) \\
&= [C \cdot (\mathrm{s} \cdot I_n - A)^{-1} \cdot B + D] \cdot \mathcal{L}[\check{u}](s) \\
&= G(s) \cdot \mathcal{L}[\check{u}](s)
\end{aligned}$$

Transfer function

$$\mathcal{L}\,[\check{y}](s) = G(s) \cdot \mathcal{L}[\check{u}](s)$$

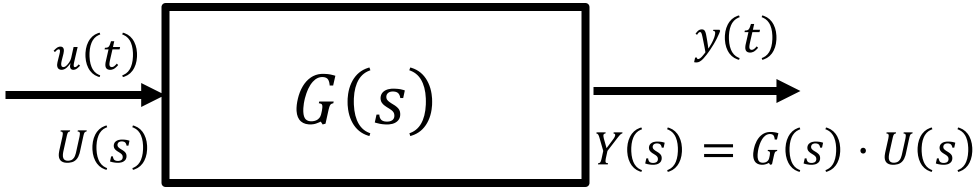
$$\frac{\mathcal{L}[\check{y}](s)}{\mathcal{L}[\check{u}](s)} = G(s) = C(sI_n - A)^{-1}B + D$$

Transfer function

The ratio between the Laplace transform of the forced output movement and its corresponding input of a continuous time LTI system is a function of s, called *transfer function*, that does not depend on the specific input, but only on the system.

Meaning of the transfer function

- The transfer function of a continuous time LTI system allows us to compute the forced output movement using a multiplication



- Therefore, the forced output movement can be computed by computing an inverse Laplace transform.

- It is an **external representation** of an LTI system.

Example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a \cdot d - b \cdot c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad D = 0$$

Compute the transfer function:

$$G(s) = C(sI_n - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \left(\frac{1}{s^2 + 2s + 1} \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad D = 0$$

Compute the transfer function:

$$G(s) = \frac{1}{s^2 + 2s + 1} \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + 2s + 1} \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} s \\ 1 \end{bmatrix}$$

$$= \frac{s+2}{s^2 + 2s + 1}$$

Example

Now we can ignore the matrices A, B, C and D and focus on the transfer function:

$$G(s) = \frac{s+2}{s^2+2s+1}$$

Compute the forced movement with a step input:

$$u(t) = step(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\,[u](s) = \frac{1}{s}$$

Therefore:

$$\mathcal{L}\,[y](s) = G(s) \cdot \mathcal{L}\,[u](s) = \frac{s+2}{s^2+2s+1} \cdot \frac{1}{s}$$

$$= \frac{s+2}{s^3+2s^2+s}$$

The free movement

- Using the transfer function, it is possible to compute the output forced movement

Stable System

↓

Free movement tends asymptotically to 0

↓

The output computed with the transfer function is asymptotically exact

Unstable System

↓

Free movement diverges to ∞

↓

The forced movement will be very different from the actual one

Transfer function interpretation

Consider a system with transfer function:

$$\frac{\mathcal{L}[\check{y}](s)}{\mathcal{L}[\check{u}](s)} = G(s)$$

And the input $u(t) = imp(t)$:

$$u(t) = imp(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \Rightarrow \quad \mathcal{L}\,[u](s) = 1$$

$$\frac{\mathcal{L}[\check{y}](s)}{1} = G(s) \quad \Rightarrow \quad \mathcal{L}\,[\check{y}](s) = G(s)$$

The transfer function is the Laplace transform of the impulse response of the system

TF structure

$$G(s) = C(sI_n - A)^{-1}B + D$$

$$(sI_n - A)^{-1} = \frac{1}{\det(sI_n - A)} \cdot K(s)$$

$K(s) \in \mathbb{R}^{n \times n}$ is the adjugate matrix of $sI_n - A$

$$G(s) = \frac{CK(s)B}{\det(sI_n - A)} + D = \frac{CK(s)B + D \cdot \det(sI_n - A)}{\det(sI_n - A)}$$

TF structure

$$G(s) = \frac{CK(s)B + D \cdot \det(sI_n - A)}{\det(sI_n - A)} = \frac{N(s)}{D(s)}$$

$$D(s) = \det(sI_n - A)$$

The denominator is the characteristic polynomial $\phi(s)$ of the matrix A

$$N(s) = CK(s)B + D \cdot \det(sI_n - A)$$

The numerator is a polynomial in s

Remarks on the denominator

$$D(s) = \phi(s) = \det(sI_n - A)$$

- It has degree $p = n$
- For the fundamental theorem of algebra, it has n roots that are called poles of the system.
- The roots correspond to the eigenvalues of A
- We can assess the stability of the system by looking at its poles: ***the system is asymptotically stable if the poles of its transfer function have negative real part:*** $Re(s_i) < 0$

Remarks on the numerator

$$N(s) = CK(s)B + D \cdot \det(sI_n - A)$$

- The roots of the numerator are called zeros of the systems.
- The polynomial $CK(s)B$ has always degree $< n$

System strictly proper → $D = 0$ → $N(s)$ has degree $m < n$

System proper → $D \neq 0$ → $N(s)$ has degree $m = n$

Simplification

- There are cases when $N(s)$ and $D(s)$ have one or more common roots that cancel out in the transfer function.

- If this happens the system has some non-observable components in the input-output relation.

- In order to understand this concept, we will use an example.

Example

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad D = 0$$

The transfer function is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a \cdot d - b \cdot c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$G(s) = C(sI_n - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} s+4 & 0 \\ -1 & s+3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+4)\cdot(s+3)} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} s+3 & 0 \\ 1 & s+4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+4)\cdot(s+3)} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ s+4 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad D = 0$$

The transfer function is:

$$G(s) = \frac{\cancel{s+4}}{\cancel{(s+4)} \cdot (s+3)}$$

$$= \frac{1}{s+3}$$

To understand what this zero-pole simplification means let's look at the state-space representation.

Example

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad D = 0$$

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A \cdot \boldsymbol{x}(t) + B \cdot u(t) \\ y(t) = C \cdot \boldsymbol{x}(t) + D \cdot u(t) \end{cases} \Rightarrow \begin{cases} \dot{x_1}(t) = -4x_1(t) \\ \dot{x_2}(t) = x_1(t) - 3x_2(t) + u(t) \\ y(t) = x_1(t) + x_2(t) \end{cases}$$

Since the transfer function computes only the forced movement:

$$x_1(0) = x_2(0) = 0$$

therefore:

$$x_1(t) = 0, \forall t \geq 0$$

for this reason, the forced movement of this system is the same as the one from:

$$\begin{cases} \dot{x_2}(t) = -3x_2(t) + u(t) \\ y(t) = x_2(t) \end{cases}$$

that has as transfer function:

$$G(s) = \frac{1}{s+3}$$

Remark: simplification

Consider the case when there are c zero-pole simplifications:

- The degree of the denominator becomes $p = n - c$
- The degree of the numerator becomes $m \leq n - c$
- $p = m$ if and only $D \neq 0$, i.e. the system is not strictly proper
- There are c eigenvalues of A that are not poles (the simplified ones)
- We can assess the stability with certainty using the transfer function if and only if $c = 0$

Static gain

Recalling that:

$$G(s) = C(sI_n - A)^{-1}B + D$$

we can note that:

$$\begin{aligned} G(0) &= C(0 \cdot I_n - A)^{-1}B + D \\ &= -CA^{-1}B + D \\ &= \mu \end{aligned}$$

where μ is the static gain of the system.

It is not defined if the denominator $D(s)$ is such that $D(0) = 0$.

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