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DI BERGAMO

Dipartimento
di Ingegneria Gestionale,
dell'Informazione e della Produzione

Lesson 6.

Movements, Equilibria, Stability of continuous-time systems

CONOTRL AND MODELING OF BIOLOGICAL SYSTEMS

MASTER DEGREE IN MEDICAL ENGINEERING

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Outline

1. Movements, equilibrium

2. LTI systems: movements, equilibrium

3. Stability

State-Space Representation

The generic state-space representation of a time-invariant continuous time nonlinear dynamical system

$$\begin{cases}\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(\boldsymbol{t}), u(t)) & \longrightarrow \text{State Equation} \\ y(t) = g\big(\boldsymbol{x}(\boldsymbol{t}), u(t)\big) & \longrightarrow \text{Output Equation}\end{cases}$$

$$\boldsymbol{x}(t) \in \Re^{\boldsymbol{n}}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x_0} \longrightarrow \text{Initial state}$$

State variables are internal variables ($\boldsymbol{x}(\boldsymbol{t})$) of the system whose knowledge at the time t_0 is the minimum amount of information needed to determine the output $\boldsymbol{y}(\boldsymbol{t})$ due to the **input** $\boldsymbol{u}(\boldsymbol{t})$, for all $t > t_0$

SISO → Single Input Single Output

$u(t) \in \Re$
scalar

$y(t) \in \Re$
scalar

MIMO → Multi Input Multi Output

$u(t) \in \Re^{\boldsymbol{m}}$
array

$y(t) \in \Re^{\boldsymbol{p}}$
array

State-Space Representation

When there are no input variables, the system

$$\dot{\boldsymbol{x}}(t) = f\big(\boldsymbol{x}(\boldsymbol{t})\big)$$

Is defined as **autonomous**.

When the function $f(\boldsymbol{x}, \boldsymbol{u})$ is linear in $\boldsymbol{x}(\boldsymbol{t})$ e $\boldsymbol{u}(\boldsymbol{t})$, the system is **linear time-invariant** (LTI):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with $\boldsymbol{A} \in \Re^{\boldsymbol{n},\boldsymbol{n}}$, $\boldsymbol{B} \in \Re^{\boldsymbol{n},\boldsymbol{m}}$, $\boldsymbol{C} \in \Re^{\boldsymbol{p},\boldsymbol{n}}$ e $\boldsymbol{D} \in \Re^{\boldsymbol{p},\boldsymbol{m}}$.

State and output movement

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = g(\boldsymbol{x}(t), u(t)) \end{cases}$$

Given an input function $\boldsymbol{u}(t) = \breve{\boldsymbol{u}}(t)$ (t≥ 0), and the initial condition $\boldsymbol{x}_0$, we can easily compute how state and output evolves throughout the time, for t> 0.

The functions $\breve{\boldsymbol{x}}(t)$ (t ≥ 0) and $\breve{\boldsymbol{y}}(t)$ (t ≥ 0) are respectively called **state movement** and **output movement**.

State movement

If we integrate the first equation we obtain the state movement $\breve{\boldsymbol{x}}(t)$

Output movement

Substituting the result in the second equation, we get the output movement $\breve{\boldsymbol{y}}(t)$

Example: SIR model

Infections rate

Recovered/death rate

$$
\begin{aligned}
\dot{S}(t) &= -\frac{\beta S(t) I(t)}{N} \\
\dot{I}(t) &= \frac{\beta S(t) I(t)}{N} - \gamma I(t) \\
\dot{R}(t) &= \gamma I(t)
\end{aligned}
$$

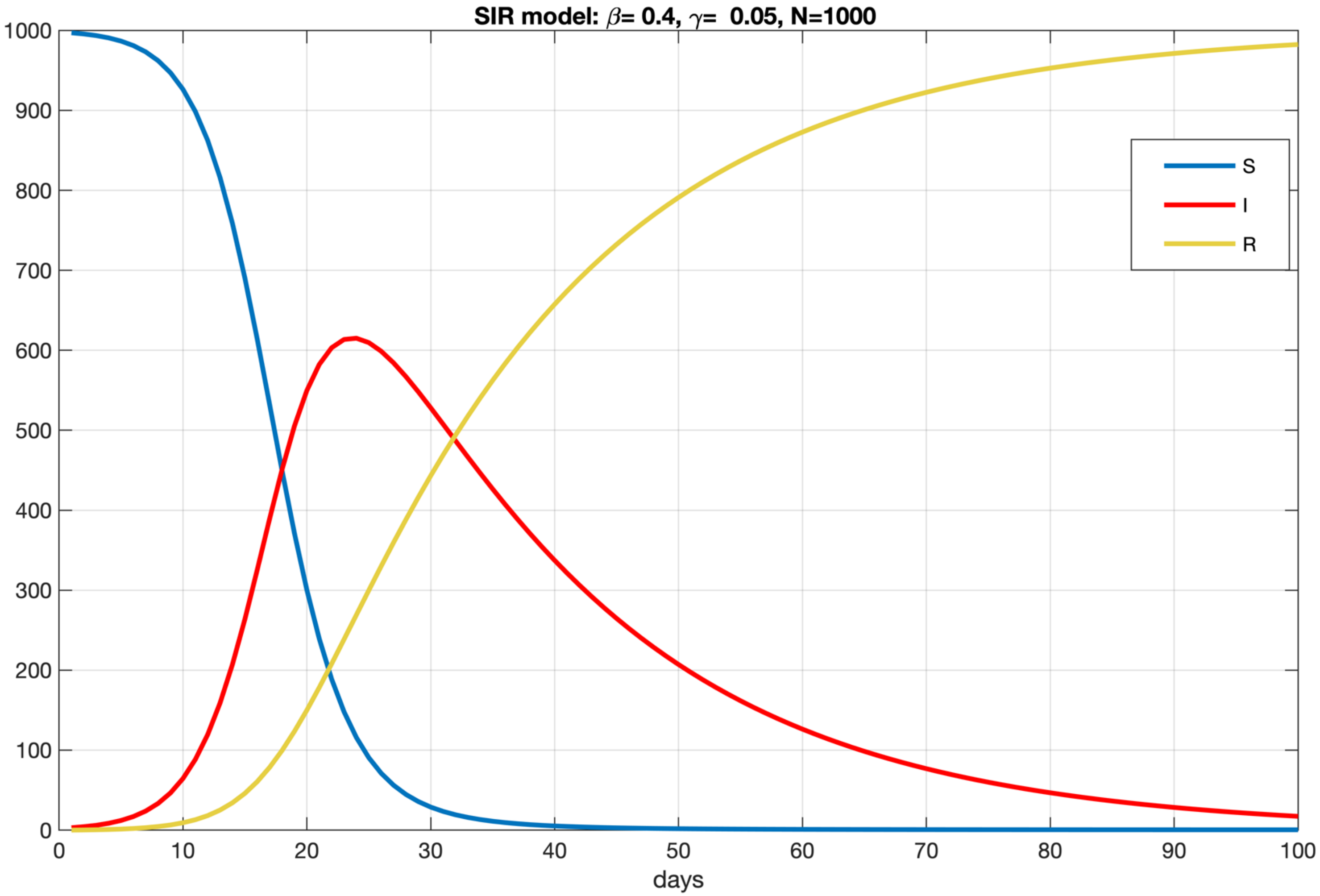
$$\boxed{\dot{S} = \frac{dS}{dt}, \quad \dot{I} = \frac{dI}{dt}, \quad \dot{R} = \frac{dR}{dt}}$$

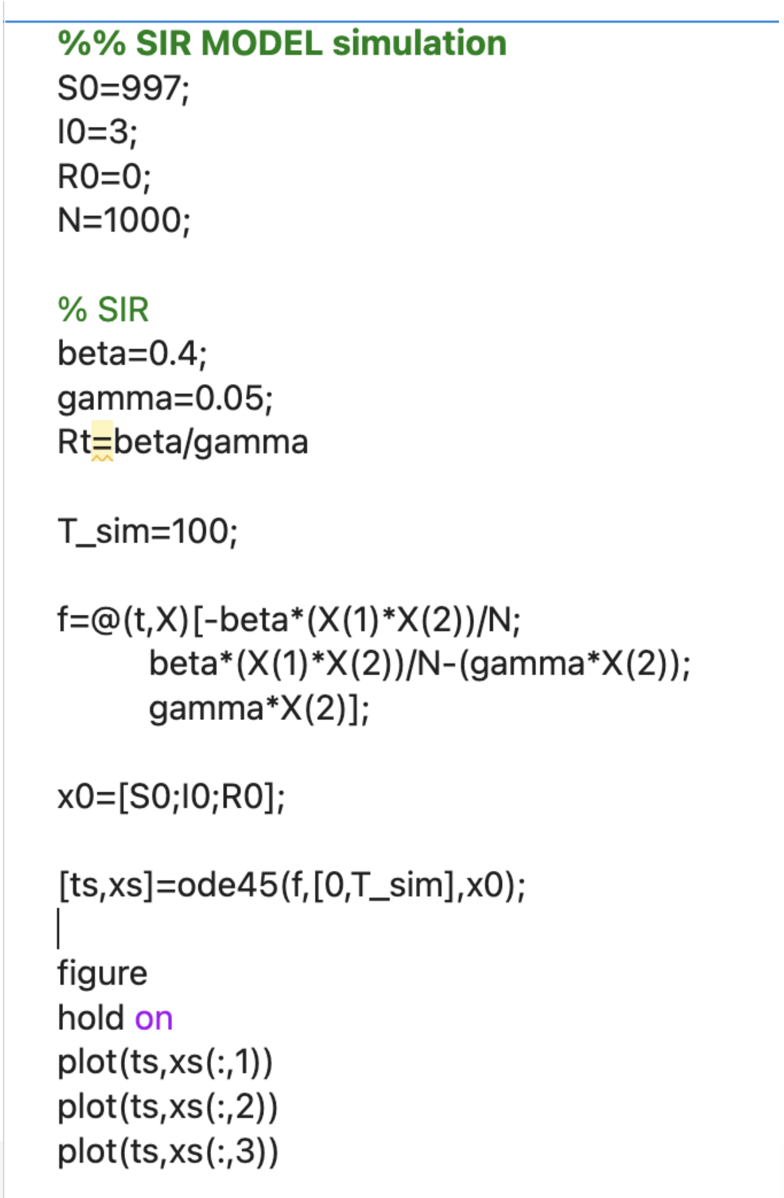
$$\boxed{R_0 = \frac{\beta}{\gamma}}$$

Basic reproduction number: the spread of an infectious disease starts if >1

Example: SIR model

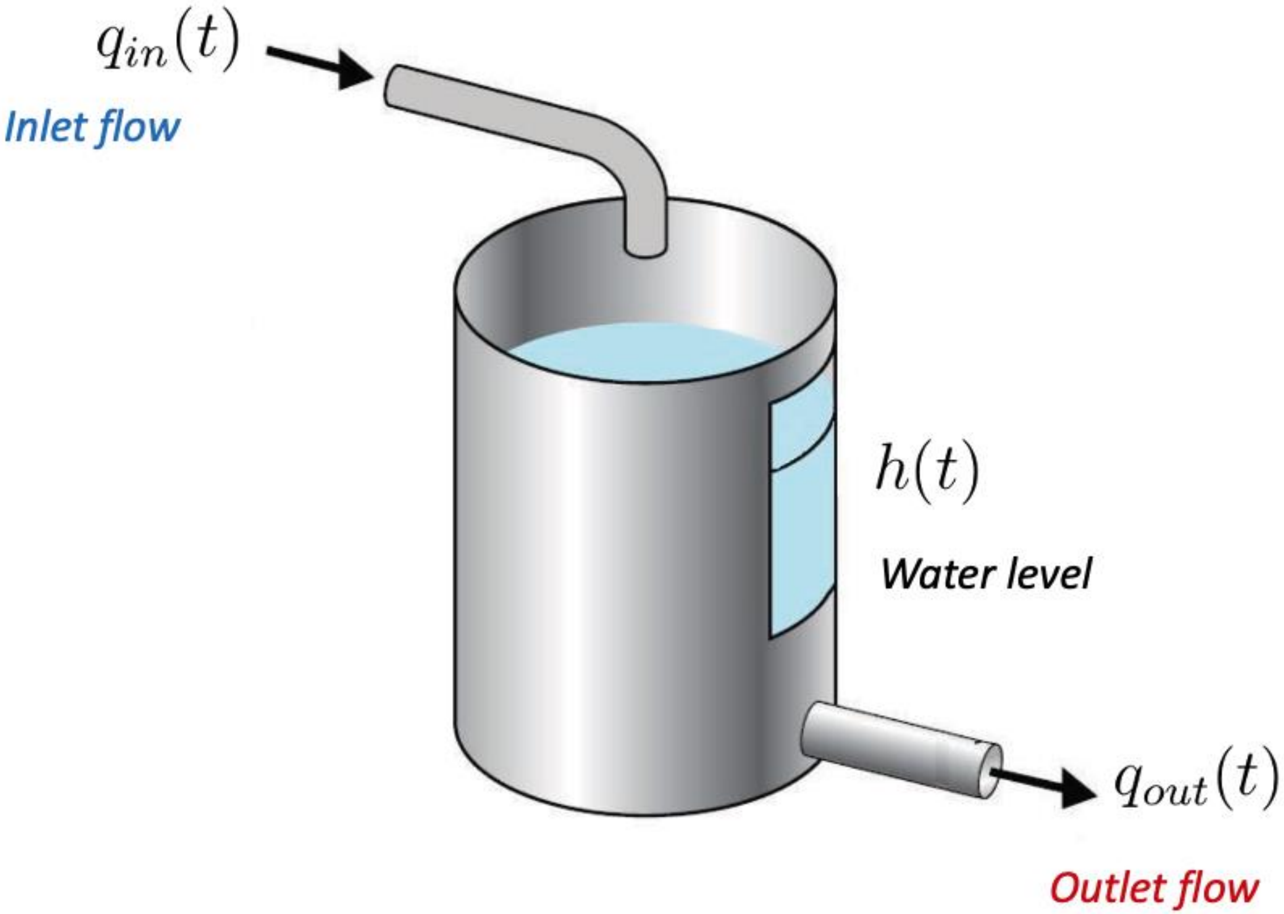
$$\beta = 0.4, \quad \gamma = 0.05$$





```
%% SIR MODEL simulation
S0=997;
I0=3;
R0=0;
N=1000;

% SIR
beta=0.4;
gamma=0.05;
Rt=beta/gamma

T_sim=100;

f=@(t,X)[-beta*(X(1)*X(2))/N;
        beta*(X(1)*X(2))/N-(gamma*X(2));
        gamma*X(2)];

x0=[S0;I0;R0];

[ts,xs]=ode45(f,[0,T_sim],x0);

figure
hold on
plot(ts,xs(:,1))
plot(ts,xs(:,2))
plot(ts,xs(:,3))
```

Example: water tank



$$
\begin{aligned}
q_{in}(t) - q_{out}(t) &= A\frac{dh(t)}{dt} \\
q_{out}(t) &= \kappa\sqrt{h(t)}
\end{aligned}
$$

$$\boxed{\frac{dh(t)}{dt} = -\frac{\kappa\sqrt{h(t)}}{A} + \frac{q_{in}(t)}{A}}$$

$h(t)$ is the solution of the differential equation

Equilibrium

If we enter constant inputs $u(t) = \bar{u}$ we obtain movements of the state and output that are also constant over time.

These movements are called **equilibrium states and outputs**. Equilibrium states must satisfy the equation $\dot{x}(t) = 0$

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = g(\boldsymbol{x}(t), u(t)) \end{cases}$$

$$u(t) = \bar{u}\,, t \geq t_0$$

$$\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = \boldsymbol{0}$$

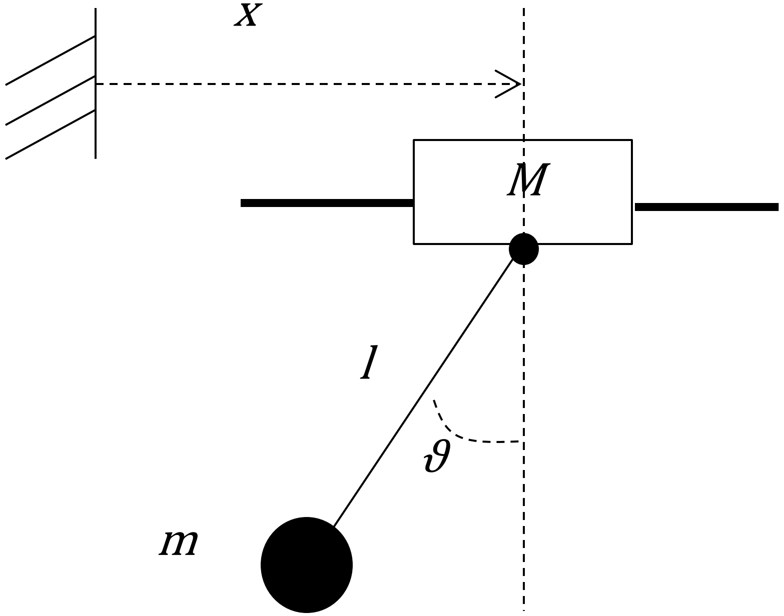
State of Equilibrium

Movement of the states $x(t) = \bar{x}$ constant over time with $u(t) = \bar{u}$

Equilibrium output

Movement of the output $y(t) = \bar{y}$ constant over time with $u(t) = \bar{u}$

Example



$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\left(\frac{u(t)}{l}\cos x_1(t) + \frac{g}{l}\sin x_1(t) + \frac{b}{ml^2}x_2(t)\right) \\ y(t) = x_1(t) \end{cases}$$

$$\boldsymbol{x}(\boldsymbol{t}) = \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} \qquad \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = \boldsymbol{0} \qquad \bar{\boldsymbol{u}} = \boldsymbol{0}$$

$$\begin{cases} 0 = \bar{x}_2 \\ 0 = -\left(\frac{\bar{u}}{l}\cos \bar{x}_1 + \frac{g}{l}\sin \bar{x}_1 + \frac{b}{ml^2}\bar{x}_2\right) \\ \bar{y} = \bar{x}_1 \end{cases}$$

$$\begin{cases} \bar{x}_2 = 0 \\ 0 = -\left(\frac{g}{l}\sin \bar{x}_1\right) \\ \bar{y} = \bar{x}_1 \end{cases}$$

Equilibria:

$$\bar{\boldsymbol{x}} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$$

Outline

1. Movements, equilibrium

2. LTI systems: movements, equilibrium

3. Stability

LTI system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = g(\boldsymbol{x}(t), u(t)) \end{cases}$$

In LTI systems, functions $f(x,u)$ and $g(x,u)$ are linear functions of $\boldsymbol{x}(\boldsymbol{t})$ e $\boldsymbol{u}(\boldsymbol{t})$, that in compact form can be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with $\boldsymbol{A} \in \Re^{\boldsymbol{n},\boldsymbol{n}}$, $\boldsymbol{B} \in \Re^{\boldsymbol{n},\boldsymbol{m}}$, $\boldsymbol{C} \in \Re^{\boldsymbol{p},\boldsymbol{n}}$ e $\boldsymbol{D} \in \Re^{\boldsymbol{p},\boldsymbol{m}}$.

This system is a **linear time-invariant** (LTI) continuous time system.

Movements

The state movement of a continuous-time LTI system can be computed using the Lagrange formula.

Given $\boldsymbol{u}(t)\ \forall t \geq 0$ and $x(0)$

$$x(t) = \boxed{e^{At}x(0)} + \boxed{\int_0^t e^{A(t-\tau)}\,Bu(\tau)d\tau}$$

Forced movement

Free movement

where $$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

Movements

$$x(t) = \underbrace{\boxed{e^{At}x(0)}}_{\text{Free movement}} + \underbrace{\boxed{\int_0^t e^{A(t-\tau)}\,Bu(\tau)d\tau}}_{\text{Forced movement}}$$

- The **free movement** only depends on the initial condition

- The **forced movement** is forced by the input applied to the system.

Output Movement

It is easy to see that

$$y(t) = \boxed{Ce^{At}x(0)} + \boxed{C\int_0^t e^{A(t-\tau)}\,Bu(\tau)d\tau + Du(t)}$$

Output
Free movement

Output
Forced movement

- ➢ The **output free movement** only depends on the initial condition
- ➢ The **output forced movement** depends only on the input applied to the system.

Equilibrium of LTI systems

Let’s assess the presence of equilibrium in continuous time LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Let's say $\dot{x}(t) = 0$ at $u(t) = \bar{u}$

$$0 = A\bar{x} + B\bar{u} \quad \Longrightarrow \quad A\bar{x} = -B\bar{u} \quad \Longrightarrow \quad \bar{x} = -A^{-1}B\bar{u}$$

$$\boxed{\det(A) \neq 0}$$

The equilibria are: $A\bar{x} = -B\bar{u}$

$$\boxed{\det(A) = 0}$$

The system $A\bar{x} = -B\bar{u}$ can have

- infinite solutions
- No solution

Outline

1. Movements, equilibrium

2. LTI systems: movements, equilibrium, stability

3. Stability

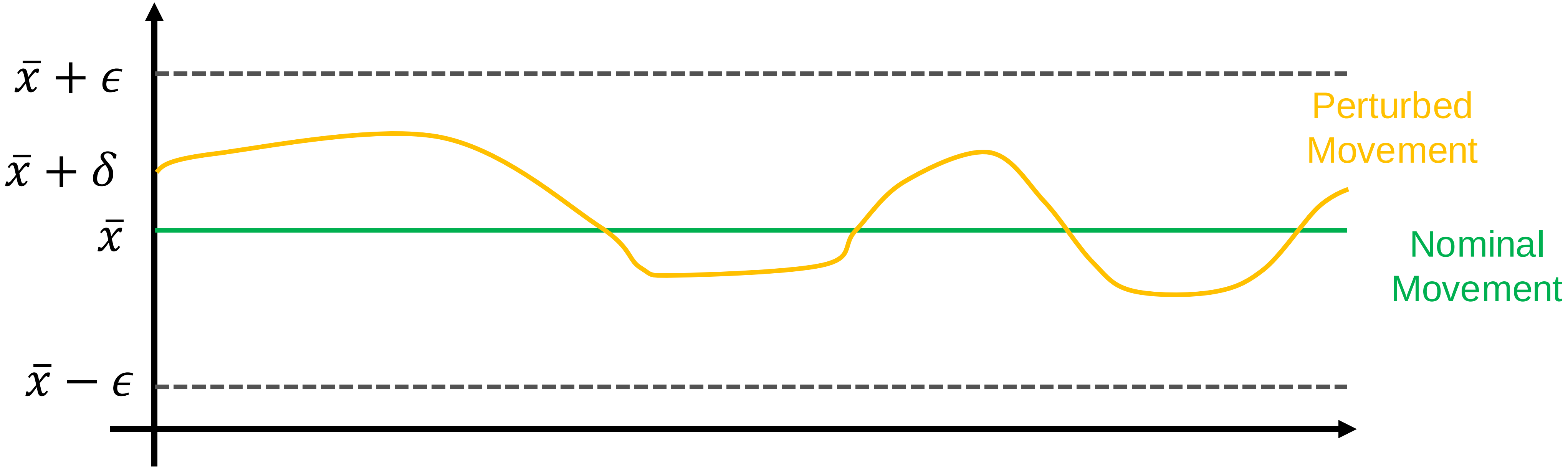
Stability

An equlibrium $\bar{\mathbf{x}}$ is said to be stable if, for each $\epsilon > 0$ there esists $\delta > 0$ such that for each initial state x_0 that satisfies:

$$\|x_0 - \bar{x}\| \leq \delta$$

It results

$$\|x(t) - \bar{x}\| \leq \epsilon \quad t \geq 0$$



Stability

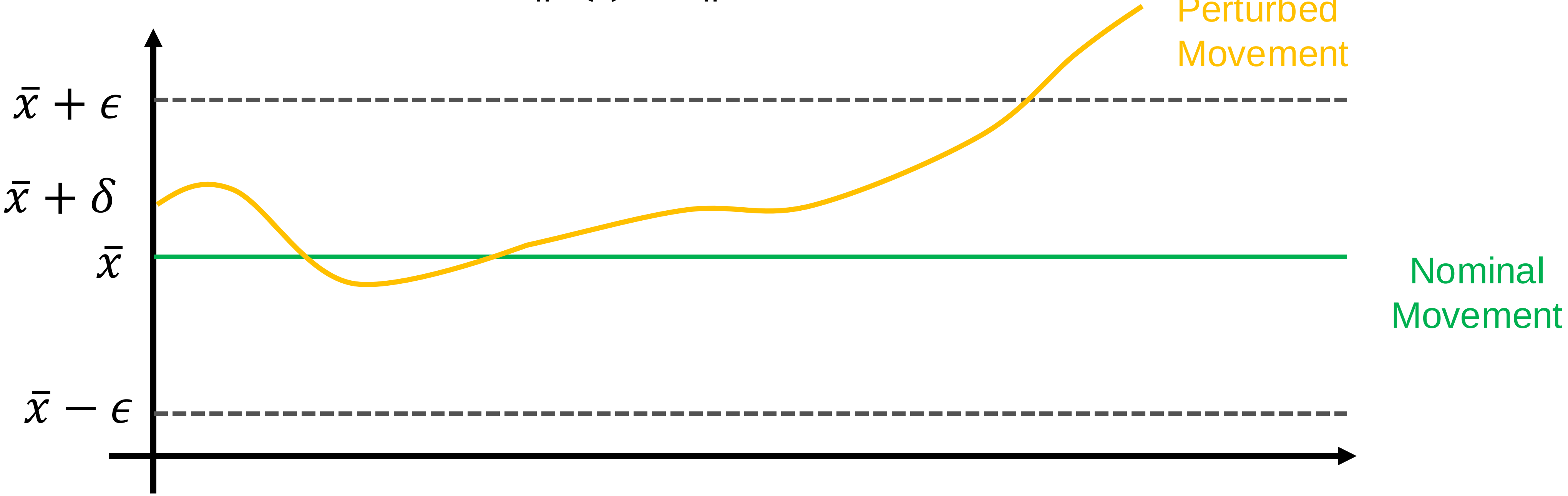
An equilibrium $\bar{\mathbf{x}}$ It is said to be **unstable** if it is not stable.

For each $\epsilon > 0$ **does not exist** $\delta > 0$ such that for each initial state x_0 that satisfies:

$$\|x_0 - \bar{x}\| \leq \delta$$

It results

$$\|x(t) - \bar{x}\| \leq \epsilon \quad t \geq 0$$



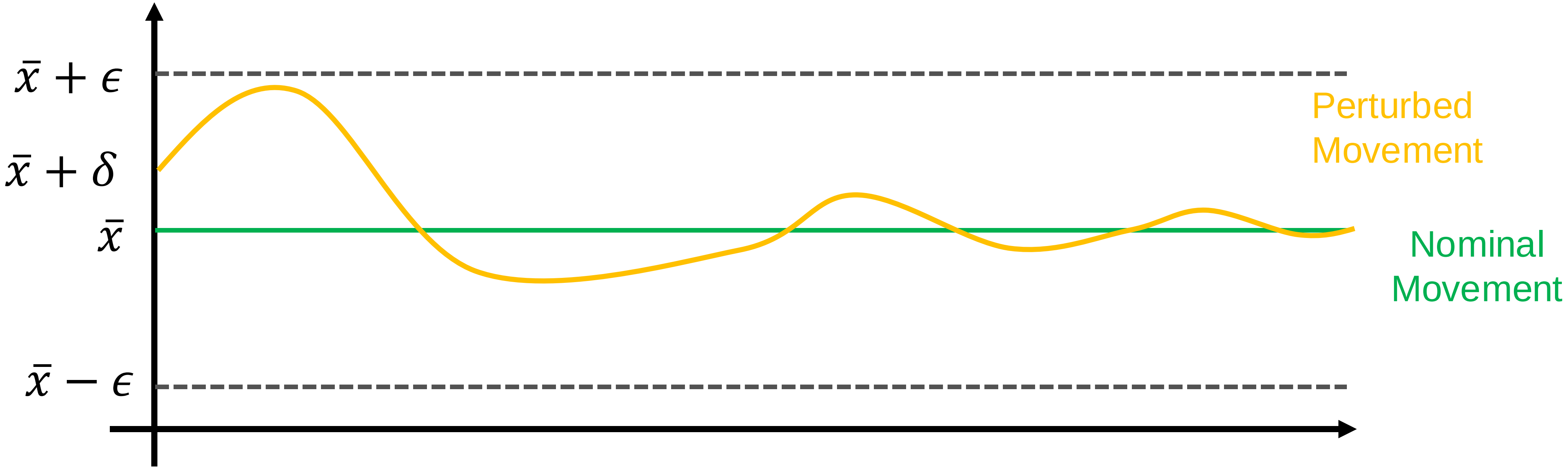
Stability

An equilibrium $\bar{\mathbf{x}}$ is said to be **asymptotically stable** if, for each $\epsilon > 0$ exists $\delta > 0$ such that for all initial states x_0 that satisfy:

$$\|x_0 - \bar{x}\| \leq \delta$$

It results

$$\|x(t) - \bar{x}\| \leq \epsilon \quad t \geq 0 \quad and \quad \lim_{t \to \infty} \|x(t) - \bar{x}\| = 0$$



Stability of LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The nominal movement of an LTI system is given by the Lagrange's formula:

$$x(t) = e^{At}x_{t0} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Assuming a perturbation of the initial condition $x_{t0} = \bar{x} + \delta_{\bar{x}}$ we get the perturbed movement:

$$\tilde{x}(t) = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + e^{At}\delta_{\bar{x}}$$

Stability of LTI systems

$$\tilde{x}(t) = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + e^{At}\delta_{\bar{x}}$$

The perturbed movement differs from the nominal movement only in that $\delta x(t) = e^{At}\delta_{\bar{x}}$.

We can therefore deduce that, for an LTI system:

- The perturbed movement does not depend on the particular state of equilibrium. We can therefore speak of the stability of the system (→ **global property**)
- The difference between the nominal and the perturbed movement depends on the values assumed by the matrix A

Stability of LTI systems

$$\tilde{x}(t) - \bar{x} = e^{At}\delta_{\bar{x}}$$

We can deduce that:

- **Asymptotically stable system** $\lim_{t\to\infty} e^{At} = 0$
- **Unstable system** e^{At} diverges with $t \to \infty$
- **Stable System** e^{At} bounded $\forall t$

Stability of LTI systems

$$\tilde{x}(t) - \bar{x} = e^{At}\delta_{\bar{x}}$$

For a first order system $A = a \in \mathbb{R}$:

- **Asymptotically stable system** $\lim_{t\to\infty} e^{at} = 0 \Longrightarrow a < 0$
- **Unstable system** e^{at} diverges with $t \to \infty \Longrightarrow a > 0$
- **Stable System** e^{at} bounded $\forall t \Longrightarrow a = 0$

What for a n order system, that is $A \in \mathbb{R}^n$?

IN-DEPTH ANALYSIS

Exponential of a square matrix

- Given a square matrix $A \in \mathbb{R}^{n\times n}$, we define its **exponential** as

$$\underset{n\times n}{e^{A}} = \sum_{i=0}^{\infty} \frac{1}{i!}\cdot (A)^{i} = I + A + \frac{1}{2}\cdot A^2 + \cdots$$

Now suppose that A is a diagonal matrix, $A = \mathrm{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$, it is possible to demonstrate that

$$e^{A} = \begin{bmatrix} e^{a_{11}} & & & \\ & e^{a_{22}} & 0 & \\ & 0 & \ddots & \\ & & & e^{a_{nn}} \end{bmatrix}$$

$$\boxed{e^{A} = \mathrm{diag}\{e^{a_{11}}, e^{a_{22}}, \dots, e^{a_{nn}}\}}$$

IN-DEPTH ANALYSIS

Exponential of a square matrix

- Instead of imposing A diagonal, we could only suppose A to be diagonalizable. That is, A can be factorized as

$$A = T_D \cdot A_D \cdot T_D^{-1}$$

$$A_D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Let us substitute this expression inside e^A

$$e^A = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot (A)^i = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot (T_D \cdot A_D \cdot T_D^{-1})^i$$

$$= I + T_D \cdot A_D \cdot T_D^{-1} + \frac{1}{2} \cdot (T_D \cdot A_D \cdot \overbrace{T_D^{-1}) \cdot (T_D}^{T_D^{-1} \cdot T_D = I} \cdot A_D \cdot T_D^{-1}) + \cdots$$

$$= I + T_D \cdot A_D \cdot T_D^{-1} + \frac{1}{2} \cdot T_D \cdot A_D^2 \cdot T_D^{-1} + \cdots$$

$$= T_D \cdot \underbrace{\left(I + A_D + \frac{1}{2} A_D^2 + \cdots \right)}_{\text{exponential of a diagonal matrix!}} \cdot T_D^{-1} = T_D \cdot e^{A_D} \cdot T_D^{-1}$$

$$\boxed{e^A = T_D \cdot \mathrm{diag}\{e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}\} \cdot T_D^{-1}}$$

Stability depends on the sign of the eigenvalues of A

Stability theorem of LTI systems

1. A (continous time) LTI system is **asymptotically stable** if and only if all eigenvalues of matrix A have **negative real part**

$$Re(s_i) < 0, \qquad \forall i$$

2. An LTI system is **unstable** if matrix A has at least **one eigenvalue with positive real part**

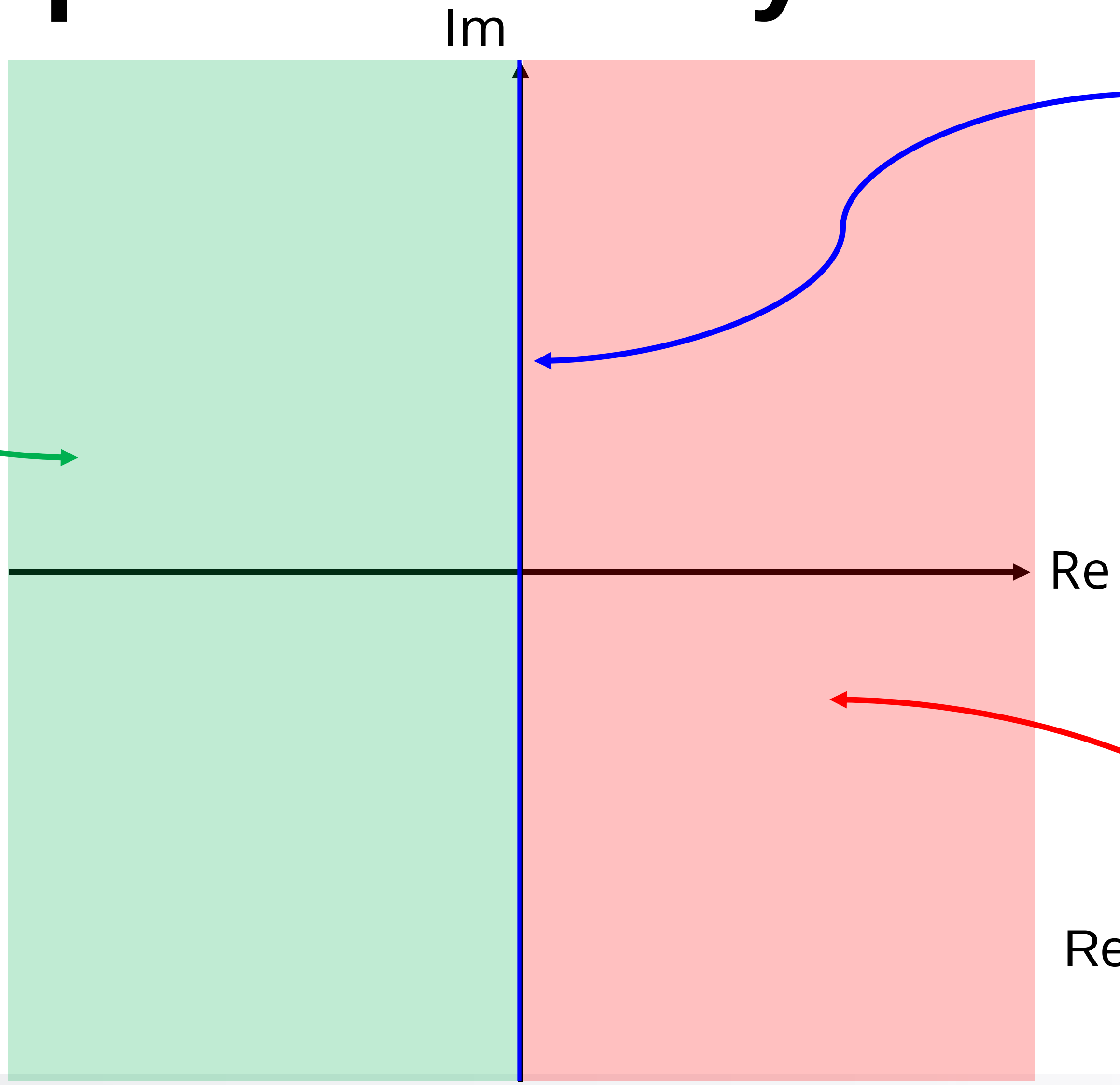
$$\exists i^*: Re\left(s_{i^*}\right) > 0$$

3. An LTI system is **stable** if matrix A has **all eigenvalues with negative real part and one null**

$$Re(s_i) < 0, \qquad \forall i$$

$$\exists!\, i^* : Re(s_{i^*}) = 0$$

Area of asymptotic stability



Stability Limit
$Re(s_i) = 0$

Region of as. stability
$Re(s_i) < 0$

Region of instability
$Re(s_i) > 0$

Properties of LTI systems

1. An as. Stable LTI system, if perturbed, tends to return to equilibrium before the perturbation.
2. At any constant input $\bar{u}$ is associated **one and only one** state of equilibrium $\bar{x}$
3. **An as. stable system is not affected by the initial conditions** (the movement of the state depends only on $u(t)$)

Proof:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{At}x(0) + \lim_{t\to\infty}\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \lim_{t\to\infty}\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Goes to 0

Properties of LTI systems

4. With zero input, the movement of the state tends asymptotically to zero.

$$\lim_{t\to\infty} x(t) = \boxed{\lim_{t\to\infty} e^{At} x(0)} + \lim_{t\to\infty} \int_0^t e^{A(t-\tau)} B \boxed{u(\tau)} d\tau = 0$$

Goes to 0

$u(\tau)$=0

5. With $u(t) = \bar{u}$ the output of an as. stable system tends to the stationary value $\bar{y}$.

6. **If the input is bounded, the output of an as. Stable LTI system will also be bounded**

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