UNIVERSITÀ
DEGLI STUDI
DI BERGAMO

Dipartimento
di Ingegneria Gestionale,
dell'Informazione e della Produzione

Lesson 5.

Frequency Response

CONTROL AND MODELING OF BIOLOGICAL SYSTEMS

MASTER DEGREE IN MEDICAL ENGINEERING

TEACHER
Antonio Ferramosca

PLACE
University of Bergamo

Sampled sine wave

The sampled value of a sine wave with sampling period T_S are:

$$s(t) = \underset{\text{Amplitude}}{A} \cdot \sin(2\pi \underset{\text{Frequency}}{f_0} \cdot t \cdot T_S + \underset{\text{Phase}}{\varphi})$$

With sampling period T_S the Nyquist frequency is:

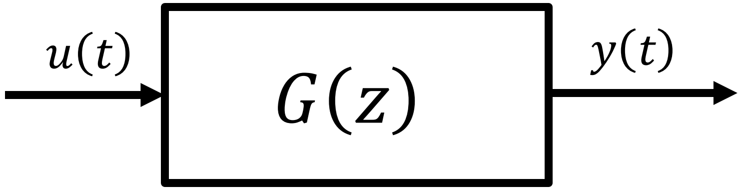
$$f_{Nyq} = \frac{f_S}{2} = \frac{1}{2 \cdot T_S}$$

In order to correctly sample, the sampled sine wave has to respect the *Nyquist theorem*:

$$f_0 \leq f_{Nyq} = \frac{f_S}{2} = \frac{1}{2 \cdot T_S}$$

Frequency response theorem

Consider an asymptotically stable LTI system with transfer function $G(z)$.



If the input signal is: $u(t) = A \cdot \sin(2\pi f_0 T_S \cdot t + \varphi)$

Then the output signal will be:

$$y(t) = \tilde{y}(t) + \bar{A} \cdot \sin(2\pi f_0 T_S \cdot t + \bar{\varphi})$$

where:

$$\lim_{t\to\infty} \tilde{y}(t) = 0 \qquad \bar{A} = A \cdot \left|G\left(e^{j\cdot 2\pi f_0 T_S}\right)\right| \qquad \bar{\varphi} = \varphi + \angle G\left(e^{j\cdot 2\pi f_0 T_S}\right)$$

Transient effect **System Gain effect** **System phase shift**

Transient

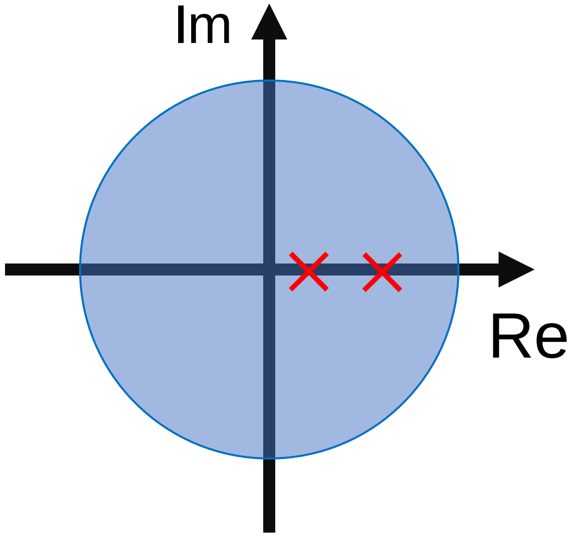
- The signal $\tilde{y}(t)$ is called transient and becomes negligible after an amount of time that depends on the position of the poles.

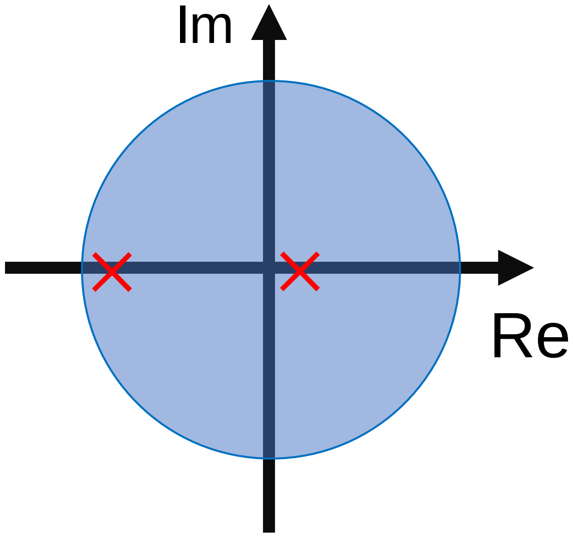
- Slower poles are close to the unitary circle and faster poles are close to zero.

- The transient time length depends on the **slowest pole** of the system.

- We won’t show why this is the case, but it’s important to understand when a system has longer transient.

Examples





$$G_1(z) = \frac{1}{(z-0.1)\cdot(z-0.5)}$$

$$G_2(z) = \frac{1}{(z+0.9)\cdot(z-0.05)}$$

G_2 has a longer transient with respect to G_1 because the pole $p = 0.9$ is closer to 1 with respect to the pole $p = 0.5$ of G_1.

Frequency response function

After the transient the output of a sine wave is:

$$\begin{aligned} y(t) &= \bar{A} \cdot \sin(2\pi f_0 T_S \cdot t + \bar{\varphi}) \\ &= A \cdot \left|G\left(e^{j\cdot 2\pi f_0 T_S}\right)\right| \cdot \sin\left(2\pi f_0 T_S \cdot t + \varphi + \angle G\left(e^{j\cdot 2\pi f_0 T_S}\right)\right) \end{aligned}$$

Defining the ***FRF (Frequency Response Function)*** as:

$$H_{T_S}(f) = G\left(e^{j\cdot 2\pi T_S \cdot f}\right)$$

we can write:

$$y(t) = A \cdot \left|H_{T_S}(f_0)\right| \cdot \sin\left(2\pi f_0 \cdot t + \varphi + \angle H_{T_S}(f_0)\right)$$

Remarks

- The output of an LTI with a sine wave input, after the transient, is another sine wave with the same frequency.

- The FRF depends only on the system and the sampling period/frequency

- A certain FRF can be used only for a certain sampling frequency and only for a sine wave that respects the Nyquist theorem:

$$f_0 < f_{Nyq} = \frac{f_S}{2} = \frac{1}{2T_S}$$

Example

$$T_S = 0.1$$

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

Compute the output of the system with input:

$$u(t) = \sin(\pi T_S \cdot t)$$

$$u(t) = A \cdot \sin(2\pi f_0 T_S \cdot t + \varphi)$$

Check the Nyquist criterion:

$$f_0 = 0.5Hz$$

$$f_{Nyq} = \frac{1}{2 \cdot T_S} = 5Hz$$

Since $0.5Hz \leq 5Hz$ *the condition is respected*.

Example

$$\boxed{T_S = 0.1}$$

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

Compute the output of the system with input: $u(t) = \sin(\pi T_S \cdot t)$

Check the stability: $D(z) = z^2 - 1.8z + 0.9 = 0$

$$\lambda_{1,2} = \frac{1.8 \pm \sqrt{1.8^2 - 4 \cdot 1 \cdot 0.9}}{2 \cdot 1}$$

$$= \frac{1.8 \pm \sqrt{-0.36}}{2}$$

$$= 0.9 \pm j \cdot 0.3 \Rightarrow |\lambda_{1,2}| = \sqrt{0.9^2 + 0.3^2} = 0.9487 < 1 \Rightarrow$$

Asymptotically stable

Example

$$T_S = 0.1$$

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

Compute the output of the system with input:

$$u(t) = \sin(\pi T_S \cdot t)$$

The FRF is:

$$H_{0.1}(f) = G\left(e^{j\cdot 2\pi T_S \cdot f}\right) = G\left(e^{j\cdot 0.2\pi\cdot f}\right)$$

$$= \frac{0.05e^{j\cdot 0.2\pi\cdot f} + 0.2}{e^{2j\cdot 0.2\pi\cdot f} - 1.8e^{j\cdot 0.2\pi\cdot f} + 0.9}$$

$$= \frac{0.05e^{j\cdot 0.2\pi\cdot f} + 0.2}{e^{j\cdot 0.4\pi\cdot f} - 1.8e^{j\cdot 0.2\pi\cdot f} + 0.9}$$

Example

$$\boxed{T_S = 0.1}$$

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

Compute the output of the system with input:

$$u(t) = \sin(\pi T_S \cdot t)$$

Therefore:

$$H_{0.1}(0.5) = \frac{0.05e^{j\cdot 0.2\pi\cdot 0.5} + 0.2}{e^{j\cdot 0.4\pi\cdot 0.5} - 1.8e^{j\cdot 0.2\pi\cdot 0.5} + 0.9}$$

$$= \frac{0.248 \cdot e^{j0.062}}{0.032 \cdot e^{j1.662}}$$

$$= 7.828 \cdot e^{-j1.600} \Rightarrow \boxed{\begin{aligned} |H_{0.1}(0.5)| &= 7.828 \\ \angle H_{0.1}(0.5) &= -1.600\text{rad} \\ &= -91.652° \end{aligned}}$$

Example

$$\boxed{T_S = 0.1}$$

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

Compute the output of the system with input:

$$u(t) = \sin(\pi T_S \cdot t)$$

Therefore, the output is:

$$y(t) = \tilde{y}(t) + |H(0.5)| \cdot \sin\left(\pi T_S \cdot t + \angle H(0.5)\right)$$

$$= \underbrace{\boxed{\tilde{y}(t)}}_{\text{Transient}} + 7.828 \cdot \sin(\pi T_S \cdot t - 1.600)$$

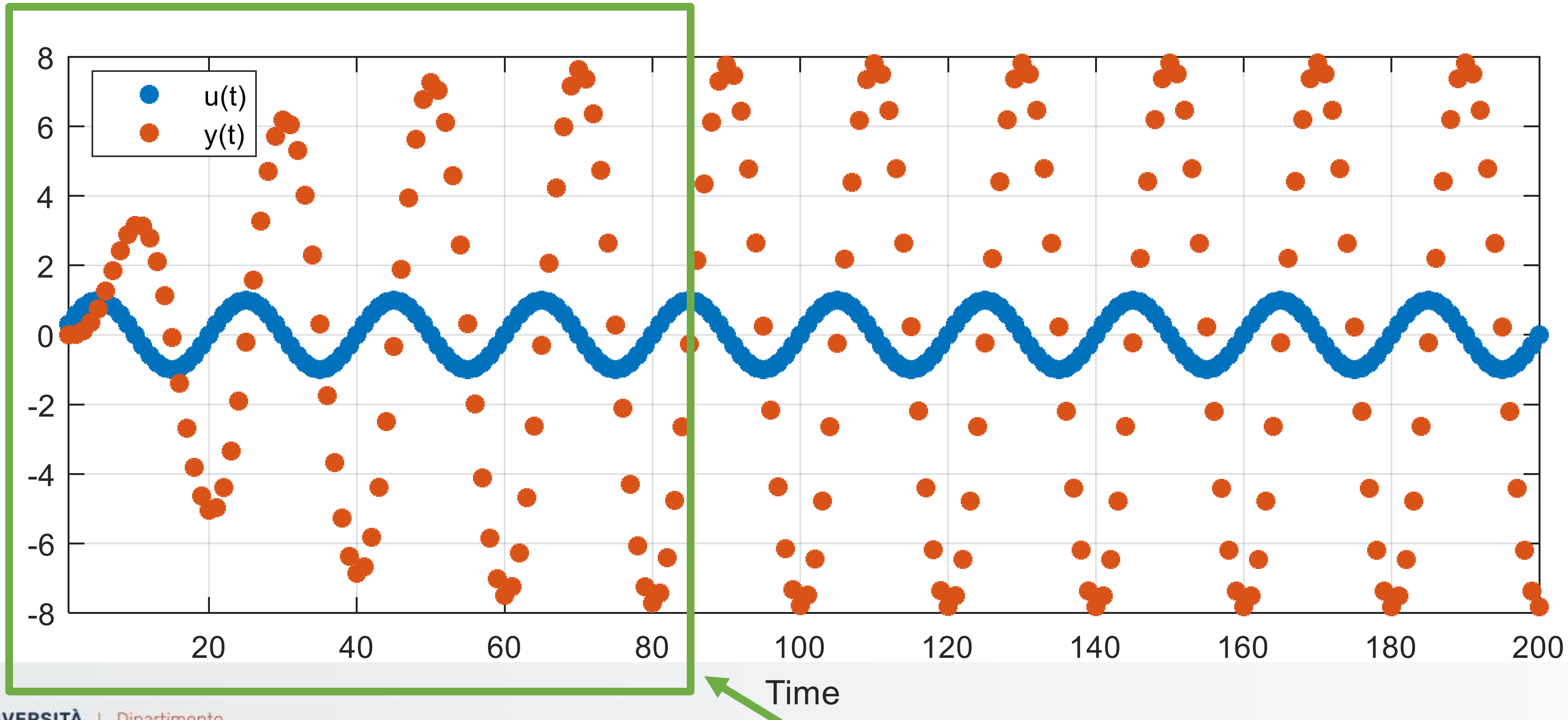
Example

$$\boxed{T_S = 0.1}$$

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

Compute the output of the system with input:

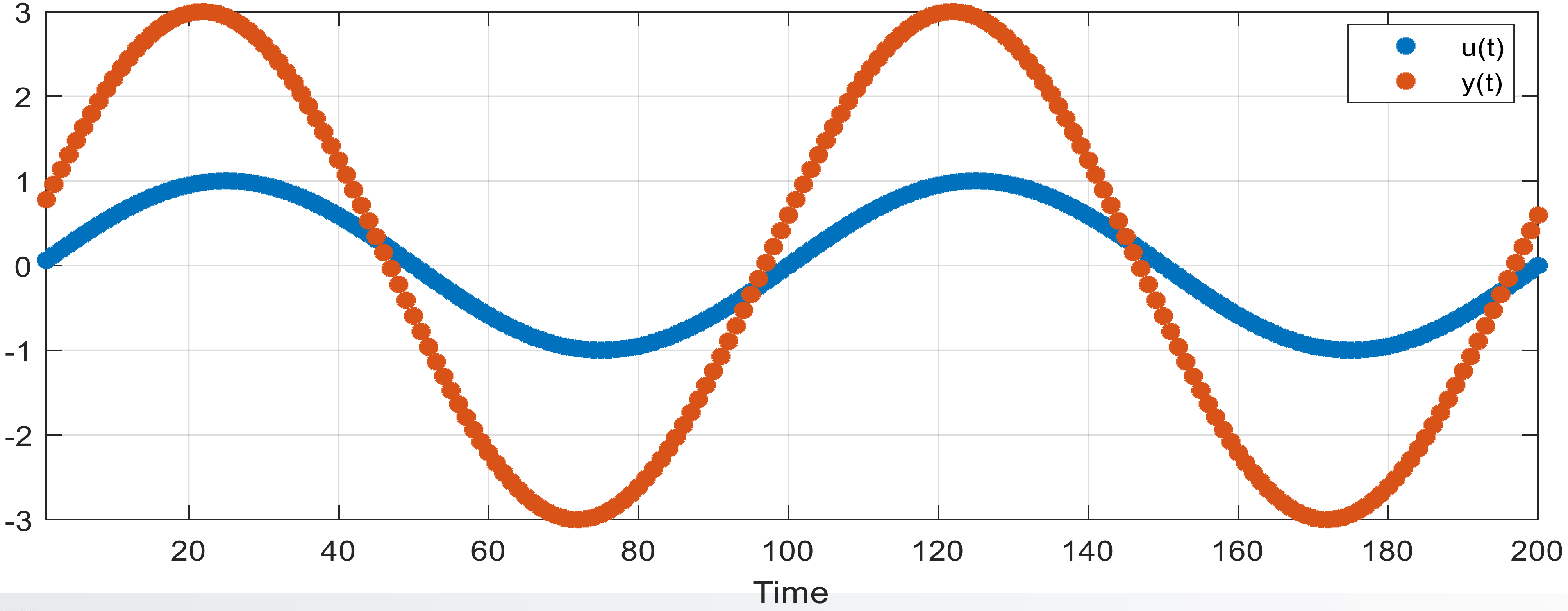
$$u(t) = \sin(\pi T_S \cdot t)$$



Transient effect

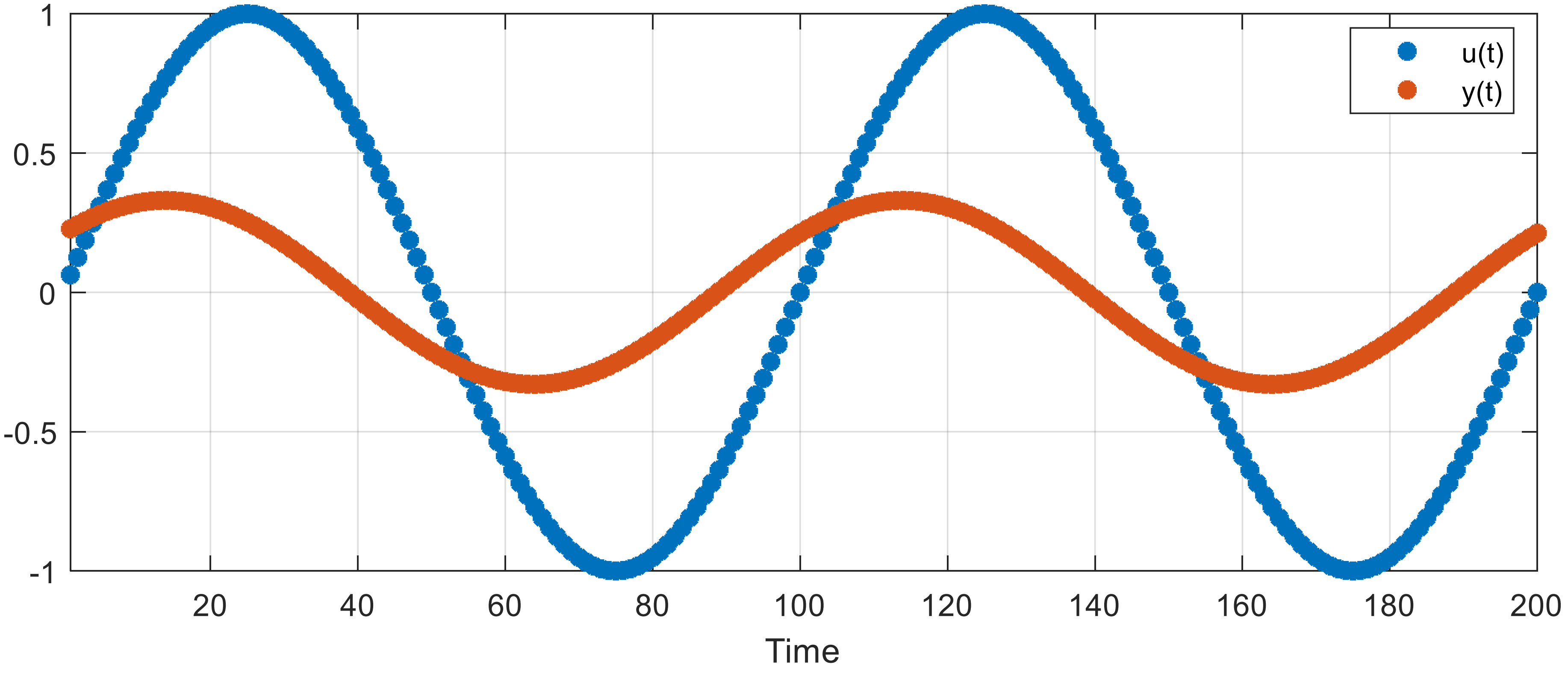
Amplification

If $\left|H_{T_S}(f_0)\right| > 1$ then the system amplifies the sine wave because its amplitude increase



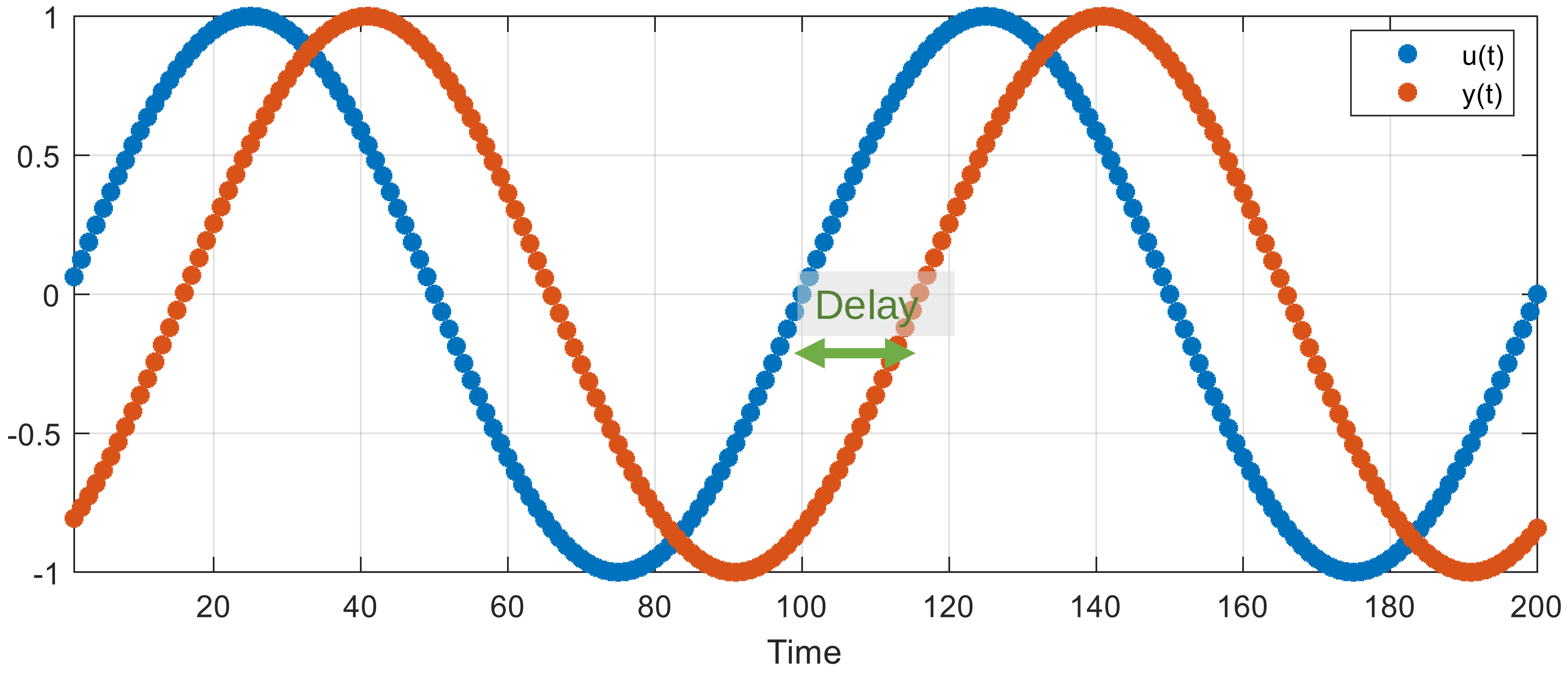
Attenuation

If $\left|H_{T_S}(f_0)\right| < 1$ then the system attenuates the sine wave because its amplitude decrease



Delay

The output sine wave is always delayed with respect to the input one.



Remarks

The FRF $H_{T_S}(f) = G\left(e^{j\cdot 2\pi T_S f}\right)$ contains all the information needed to compute the output of a sine wave input.

- The FRF is a complex function

- There are different ways to plot this function:
  - Polar graph
  - Nyquist diagram
  - Bode diagram (most used)

- These three ways to plot the FRF are equivalent and allow us to analyze the behavior of the system with sine wave inputs

Bode diagrams

The bode diagrams are composed by two graphs:

- The **magnitude graph**, that plots the **magnitude** of the system frequency response

$$\left|H_{T_s}(f)\right|$$

- The **phase graph**, that plots the **phase** of the system frequency response

$$\angle H_{T_s}(f)$$

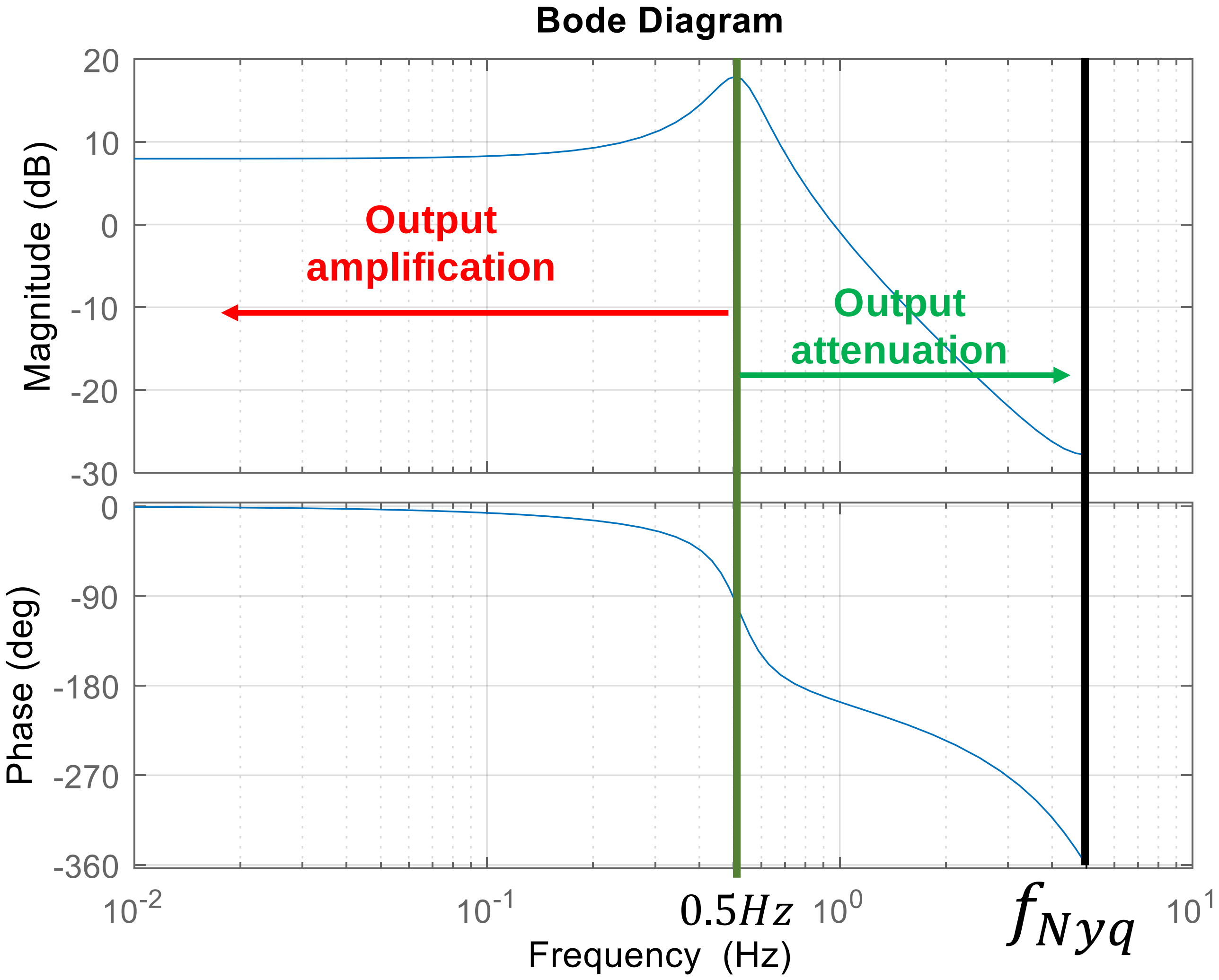
The **frequencies** are plotted in a **logarithmic scale**

The **magnitude** is expressed in **dB (decibels),** i.e. $\mathrm{dB}(f) = 20 \cdot \log\left|H_{T_s}(f)\right|$

The **phase** is usually expressed in **degrees**, but sometimes the radiants are used instead

$$G(z) = \frac{0.05z + 0.2}{z^2 - 1.8z + 0.9}$$

$$T_S = 0.1s \Rightarrow f_{Nyq} = \frac{1}{2T_S} = \frac{1}{2 \cdot 0.1} = 5$$



The green line represents the frequency used in the previous example

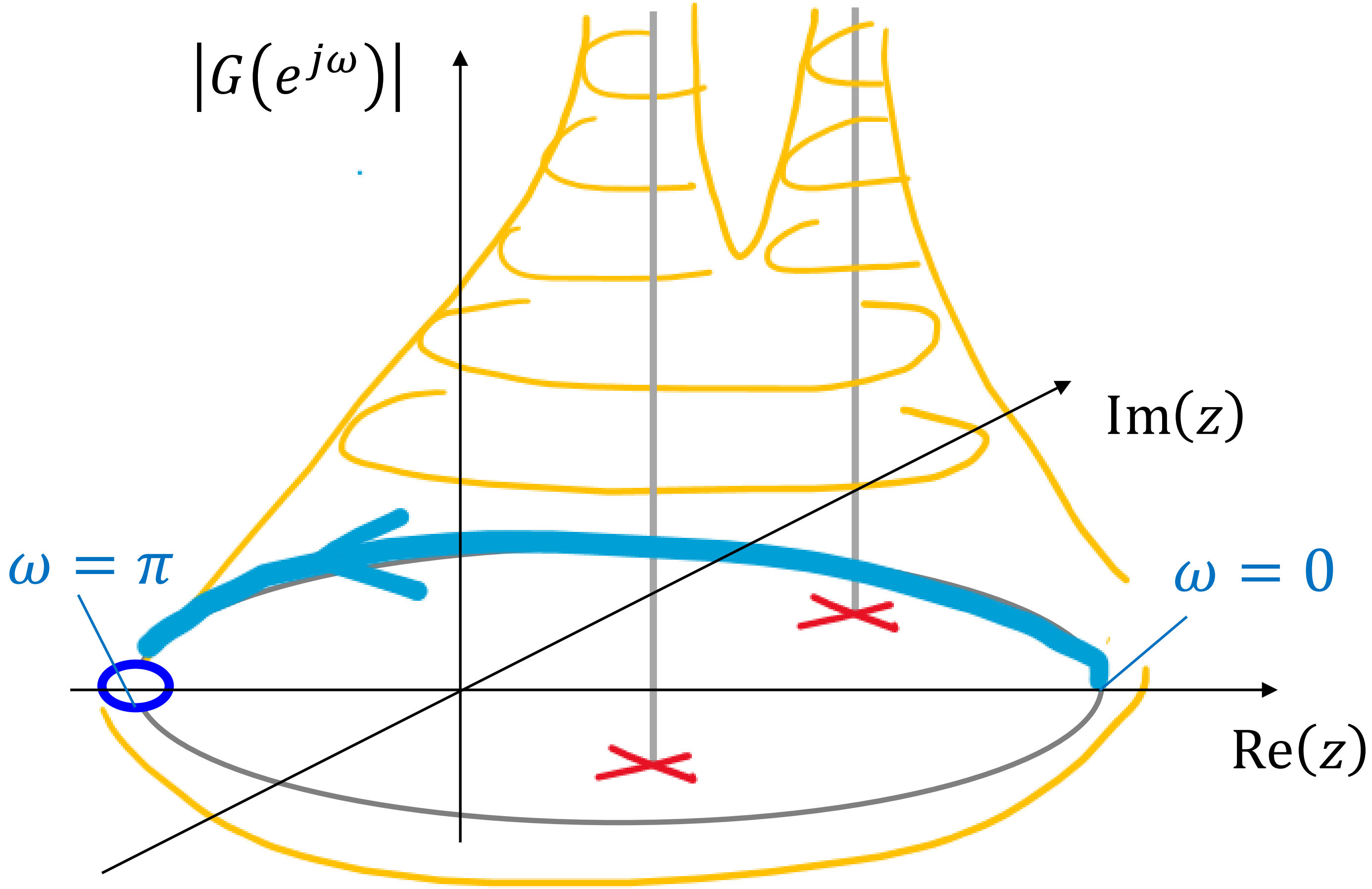
The diagram stops before the Nyquist frequency because after that the FRF has no meaning

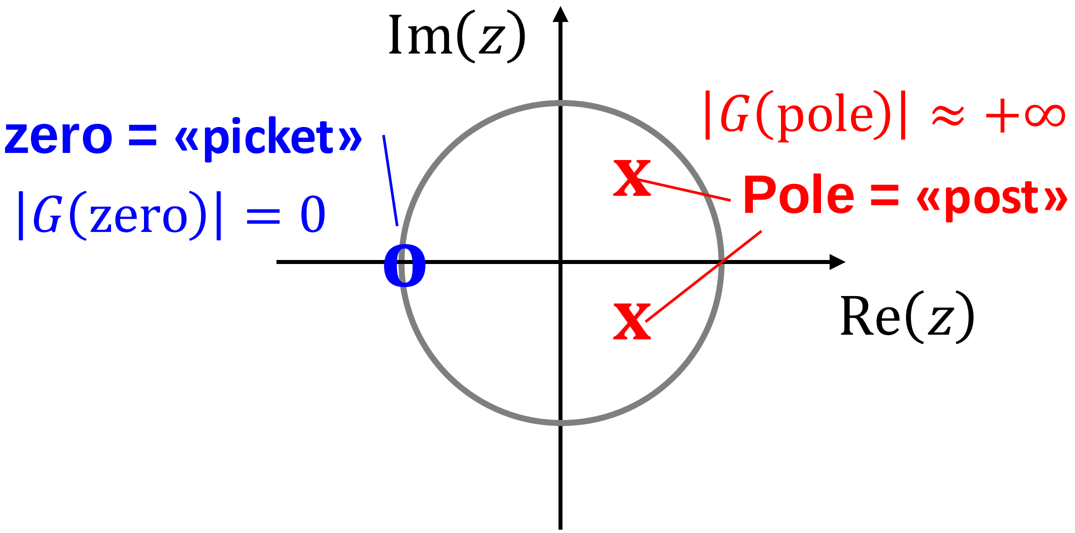
Matlab: bode( )

Remark

If I know the positions of **poles** and **zeros**, I may have an idea of how $\left|G\left(e^{j\omega}\right)\right|$ behave

It's like a **«circus tent»**



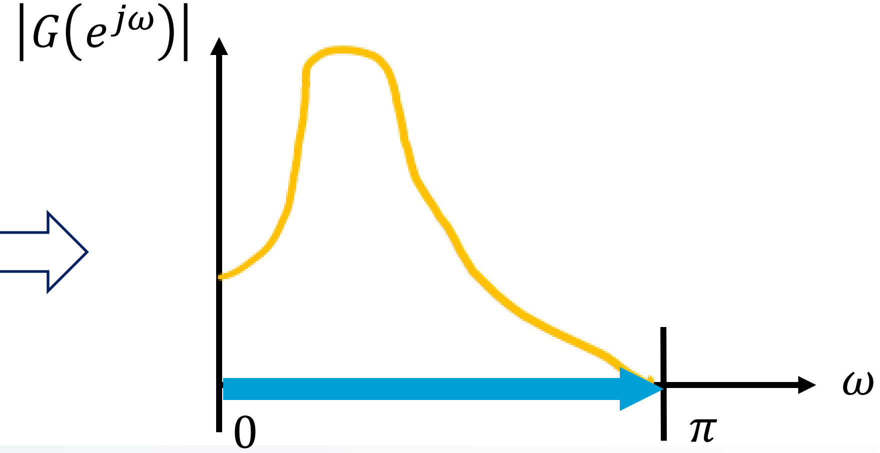


zero = «picket»

$|G(\mathrm{zero})| = 0$

$|G(\mathrm{pole})| \approx +\infty$

Pole = «post»



Remark

The discrete model depends on the sampling frequency even if it’s not always explicit.

For example, consider the water tank model that we saw in the first lesson:

$$\begin{cases} x(t+1) = x(t) + \dfrac{\Delta t}{A} \cdot \big(u(t) - k \cdot x(t)\big) \\ y(t) = k \cdot x(t) \end{cases}$$

Δt is the sampling time and by changing it the coefficient of the FRF changes.

The input should be sampled with the same frequency used to define the system.