UNIVERSITÀ
DEGLI STUDI
DI BERGAMO

Dipartimento
di Ingegneria Gestionale,
dell'Informazione e della Produzione

Lesson 3.

Movements, Equilibria, Stability

CONOTRL AND MODELING OF BIOLOGICAL SYSTEMS

MASTER DEGREE IN MEDICAL ENGINEERING

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Outline

1. Movements, equilibrium
2. Stability
3. LTI systems: movements, equilibrium, stability
4. Linearization
5. Continuous time systems

Outline

1. **Movements, equilibrium**
2. Stability
3. LTI systems: movements, equilibrium, stability
4. Linearization
5. Continuous time systems

State and output movements

$$
\begin{aligned}
x(t+1) &= f(x(t), u(t)), \quad x(0) = x_0 \\
y(t) &= g(x(t), u(t))
\end{aligned}
$$

Given an input function $\boldsymbol{u}(t) = \breve{\boldsymbol{u}}(t)$ (t ≥ 0), and the initial condition $\boldsymbol{x}_0$, we can easily compute how state and output evolves throughout the time, for t> 0.

The functions $\breve{\boldsymbol{x}}(t)$ (t ≥ 0) and $\breve{\boldsymbol{y}}(t)$ (t ≥ 0) are respectively called **state movement** and **output movement**.

$\breve{\boldsymbol{x}}(t)$, t ≥ 0 is the state movement corresponding to the input $\breve{\boldsymbol{u}}(t)$

$\breve{\boldsymbol{y}}(t)$, t ≥ 0 is the output movement corresponding to the input $\breve{\boldsymbol{u}}(t)$

Such movements can be computed iteratively.

State and output movements

Let the input function $\boldsymbol{u}(t) = \breve{\boldsymbol{u}}(t)$ (t ≥ 0), be given by the sequence:

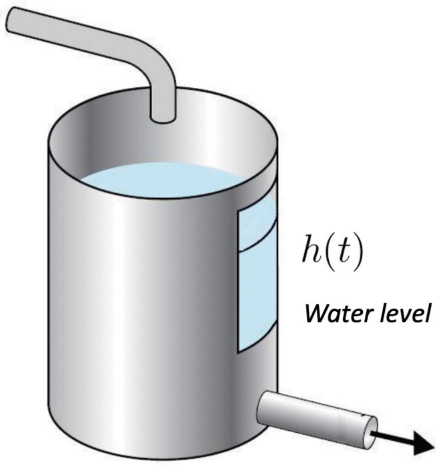
$$\breve{\mathbf{u}} = \{\breve{u}(0), \breve{u}(1), \breve{u}(2), \ldots., \breve{u}(t), \breve{u}(t+1), \ldots\}$$

Then:

$$\begin{aligned}
\breve{x}(0) &= x_0 & \longrightarrow \quad \breve{y}(0) &= g(x_0, \breve{u}(0)) \\
\breve{x}(1) &= f(\breve{x}(0), \breve{u}(0)) & \longrightarrow \quad \breve{y}(1) &= g(\breve{x}(1), \breve{u}(1)) \\
\breve{x}(2) &= f(\breve{x}(1), \breve{u}(1)) & \longrightarrow \quad \breve{y}(2) &= g(\breve{x}(2), \breve{u}(2)) \\
\breve{x}(3) &= f(\breve{x}(2), \breve{u}(2)) & \longrightarrow \quad \breve{y}(3) &= g(\breve{x}(3), \breve{u}(3)) \\
\breve{x}(4) &= f(\breve{x}(3), \breve{u}(3)) & \longrightarrow \quad \breve{y}(4) &= g(\breve{x}(4), \breve{u}(4)) \\
&\vdots & &\vdots \\
\breve{x}(t+1) &= f(\breve{x}(t), \breve{u}(t)) & \longrightarrow \quad \breve{y}(t) &= g(\breve{x}(t), \breve{u}(t))
\end{aligned}$$

Example 1: water tank

$$
\begin{aligned}
x(t+1) &= x(t) + \frac{\Delta t}{A}\left(u(t) - \kappa\sqrt{x(t)}\right) \\
y(t) &= \kappa\sqrt{x(t)}
\end{aligned}
$$



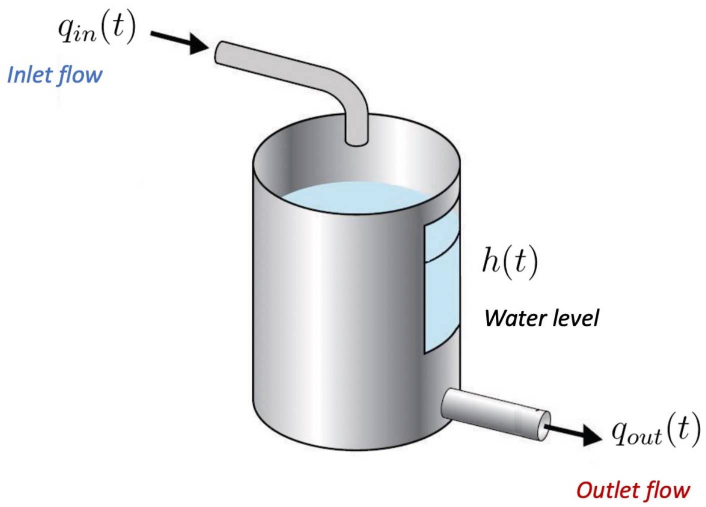
- Let: $A = 1m^2$, $\Delta t = 1s$, $k = 0.3\, {m^2}/{s}$

$$
\begin{aligned}
x(t+1) &= x(t) - 0.3\sqrt{x(t)} + u(t) \\
y(t) &= 0.3\sqrt{x(t)}
\end{aligned}
$$

- In order to find state and output movements, we need an initial condition and a control sequence:
  - Initial condition $x(0) = 4m$
  - Input $\breve{u}(t) = 0.5\, {m^3}/{s}$, $t \geq 0$

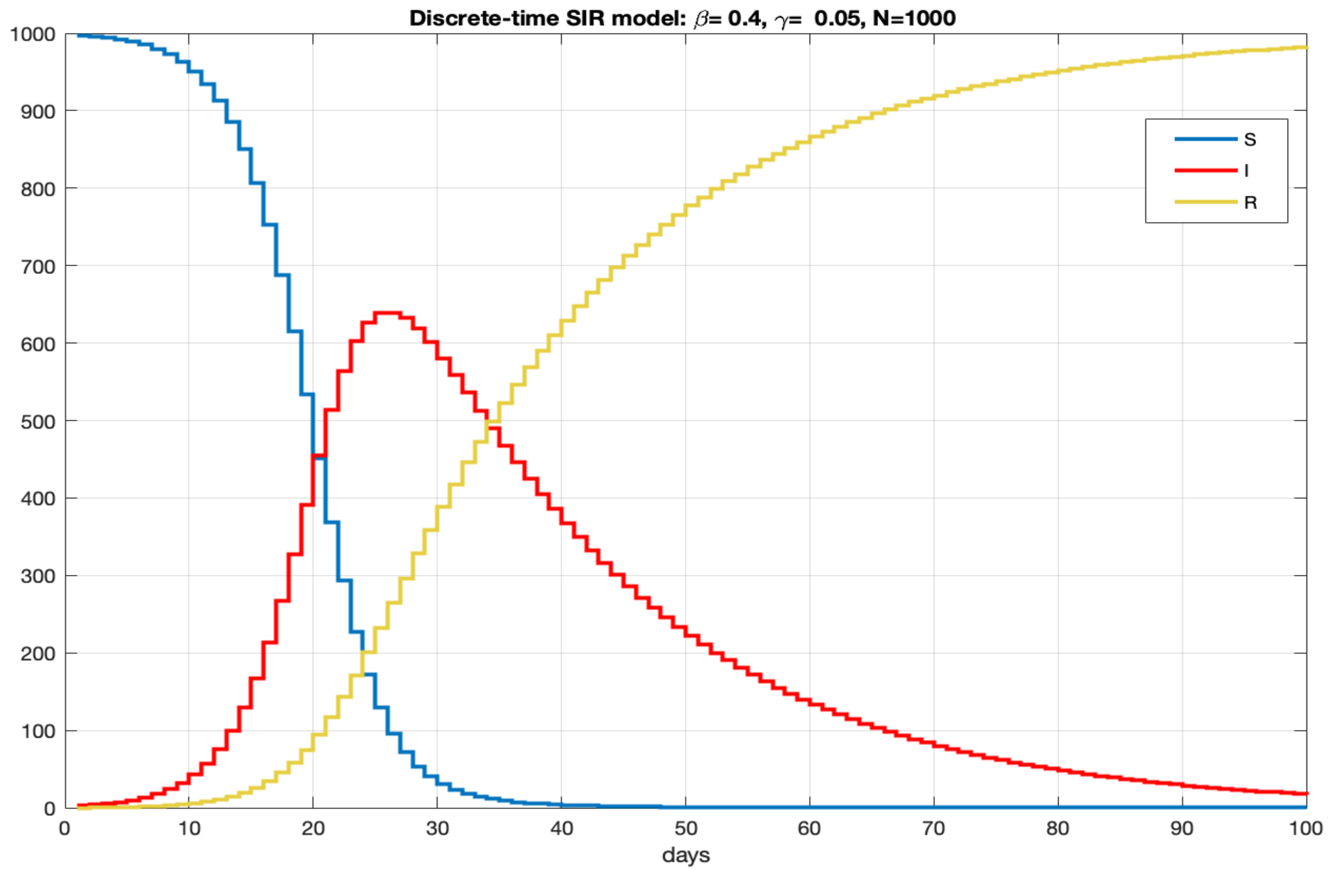
Example 1: water tank

$$
\begin{aligned}
x(t+1) &= x(t) - 0.3\sqrt{x(t)} + u(t) \\
y(t) &= 0.3\sqrt{x(t)}
\end{aligned}
$$



$$
\begin{aligned}
x(0) &= 4 \\
x(1) &= 4 - (0.3 \cdot 2) + 0.5 = 3.90 \\
x(2) &= 3.90 - (0.3 \cdot 1.97) + 0.5 = 3.81 \\
x(3) &= 3.81 - (0.3 \cdot 1.95) + 0.5 = 3.72 \\
x(4) &= 3.72 - (0.3 \cdot 1.93) + 0.5 = 3.64 \\
x(5) &= 3.64 - (0.3 \cdot 1.90) + 0.5 = 3.57 \\
&\vdots
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
y(0) &= 0.3\sqrt{4} = 0.6 \\
y(1) &= 0.3\sqrt{3.90} = 0.5925 \\
y(2) &= 0.3\sqrt{3.81} = 0.5856 \\
y(3) &= 0.3\sqrt{3.72} = 0.5786 \\
y(4) &= 0.3\sqrt{3.64} = 0.5724 \\
y(5) &= 0.3\sqrt{3.57} = 0.5668 \\
&\vdots
\end{aligned}
$$

Example 2: SIR model



$$
\begin{aligned}
S(t+1) &= S(t) - \frac{\beta S(t) I(t)}{N} \\
I(t+1) &= I(t) + \frac{\beta S(t) I(t)}{N} - \gamma I(t) \\
R(t+1) &= R(t) + \gamma I(t)
\end{aligned}
$$

$$\boxed{S(0) = 997,\ I(0) = 3,\ R(0) = 0}$$

$$\boxed{\beta = 0.4,\ \gamma = 0.05}$$

Iterations by computer simulations

Time step t =1 day

Example 2: SIR model

```matlab
%% SIR MODEL simulation

S0=997;
I0=3;
R0=0;
N=1000;

% SIR
beta=0.4;
gamma=0.05;

T=100;

S=zeros(T,1);
I=zeros(T,1);
R=zeros(T,1);

S(1)=S0;
I(1)=I0;
R(1)=R0;

for k=1:T
    S(k+1)=S(k)-beta*(S(k)*I(k))/N;
    I(k+1)=I(k)+beta*(S(k)*I(k))/N-gamma*I(k);
    R(k+1)=R(k)+gamma*I(k);
end
```

Iterations by Matlab simulations

$$
\begin{aligned}
S(t+1) &= S(t) - \frac{\beta S(t) I(t)}{N} \\
I(t+1) &= I(t) + \frac{\beta S(t) I(t)}{N} - \gamma I(t) \\
R(t+1) &= R(t) + \gamma I(t)
\end{aligned}
$$

$$\boxed{S(0) = 997,\ I(0) = 3,\ R(0) = 0}$$

$$\boxed{\beta = 0.4,\ \gamma = 0.05}$$

Equilibrium

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \quad x(0) = x_0 \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

Given a **constant** input function $\boldsymbol{u}(t) = \bar{\boldsymbol{u}}$ (t ≥ 0), the state movements will converge to an **equilibrium state and output.**

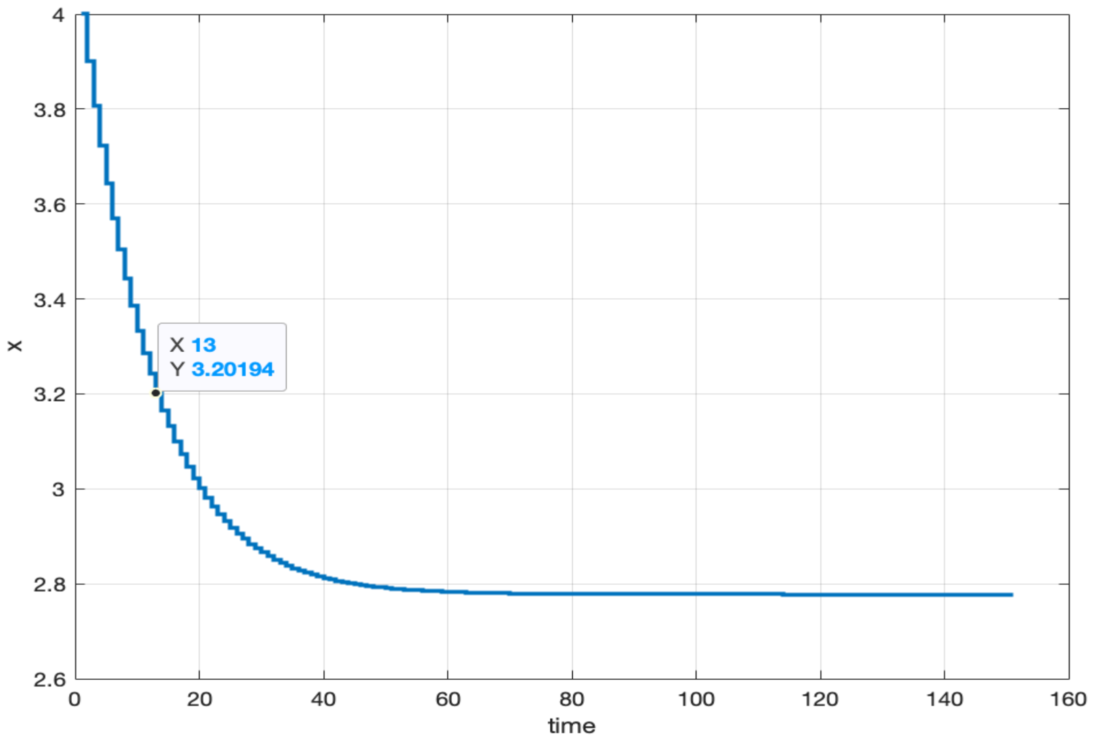
This implies that $x(t+1) = x(t)$.

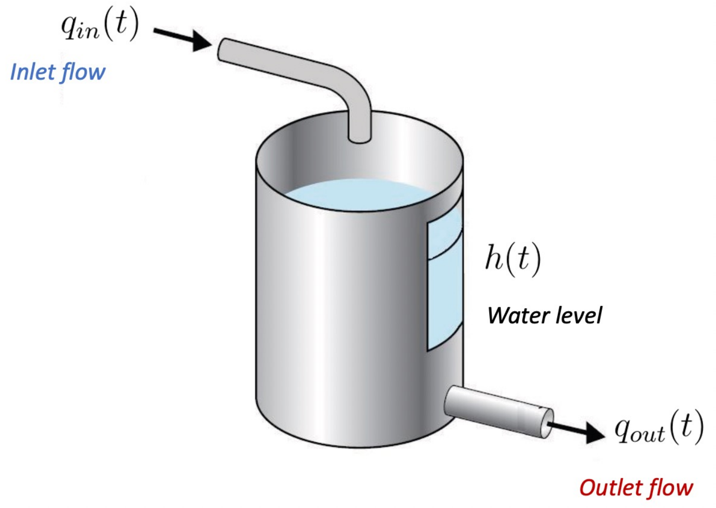
This means that functions $\boldsymbol{x}(t)$ (t ≥ 0) and $\boldsymbol{y}(t)$ (t ≥ 0) converge to a constant value $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ which is solution of the following equation

$$\boxed{\begin{aligned} \bar{x} &= f(\bar{x}, \bar{u}) \\ \bar{y} &= g(\bar{x}, \bar{u}) \end{aligned}}$$

Example 1: water tank

$$x(t+1) = x(t) - 0.3\sqrt{x(t)} + u(t)$$
$$y(t) = 0.3\sqrt{x(t)}$$





- In order to find the equilibrium state and output, we run a simulation with:
  - Initial condition $x(0) = 4m$
  - Input $\boldsymbol{u} = 0.5\ {m^3}/{s}$, k ≥ 0
- After 80 steps, the state converges to an equilibrium.

How to find the equilibrium?

Are simulations the best way to find an equilibrium?

Nop: we can also find the equilibrium analitically by solving

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{u}) \\ \bar{y} &= g(\bar{x}, \bar{u}) \end{aligned}$$

$$\begin{aligned} \bar{x} &= \bar{x} - 0.3\sqrt{\bar{x}} + \bar{u} \\ \bar{y} &= 0.3\sqrt{\bar{x}} \end{aligned} \longrightarrow \begin{aligned} \bar{x} &= \left(\frac{\bar{u}}{0.3}\right)^2 \\ \bar{y} &= \bar{u} \end{aligned}$$

- Taking into account that $\bar{\boldsymbol{u}} = 0.5\ {}^{m^3}/_{s}$, then

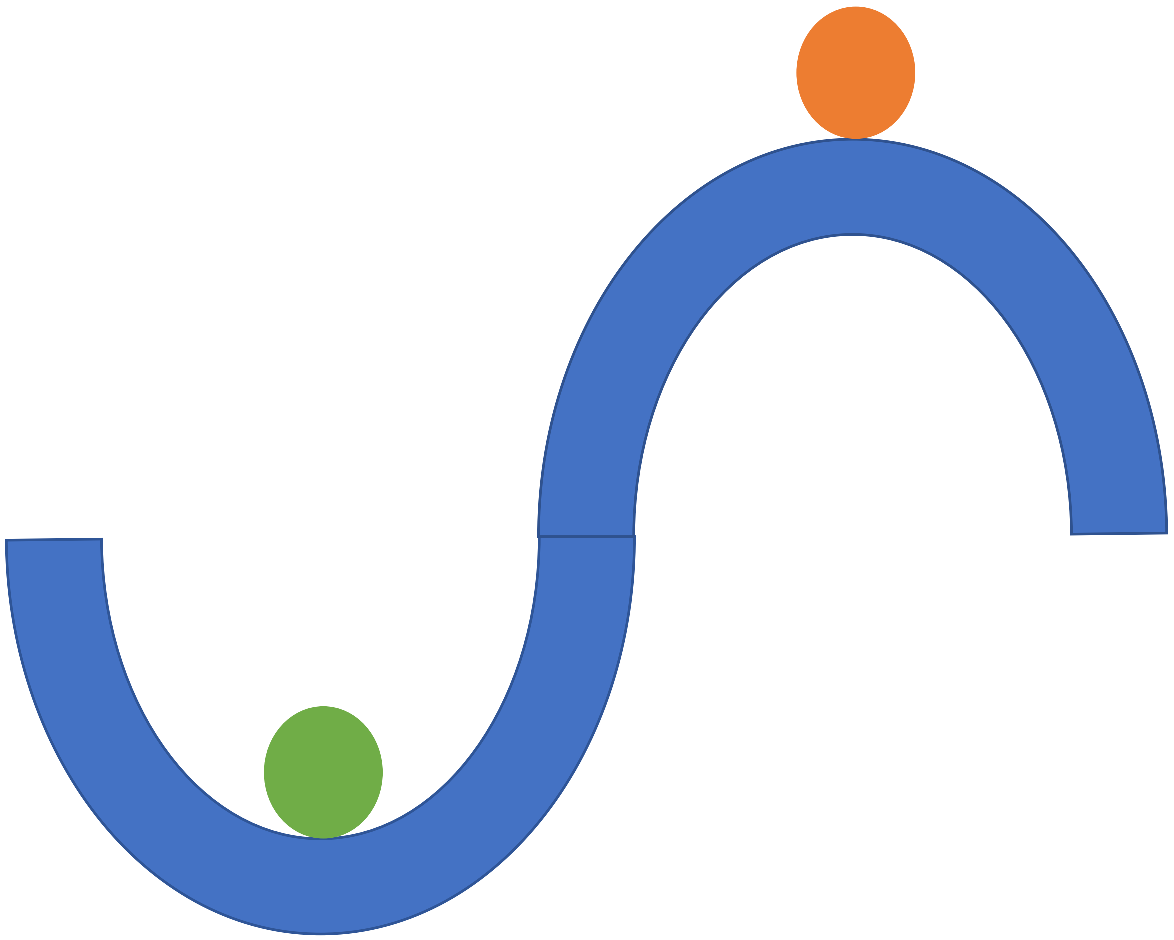
$$\begin{aligned} \bar{x} &= 2.7778 \\ \bar{y} &= 0.5 \end{aligned}$$

At the equilibrium the outlet flow is equal to the inlet flow

Outline

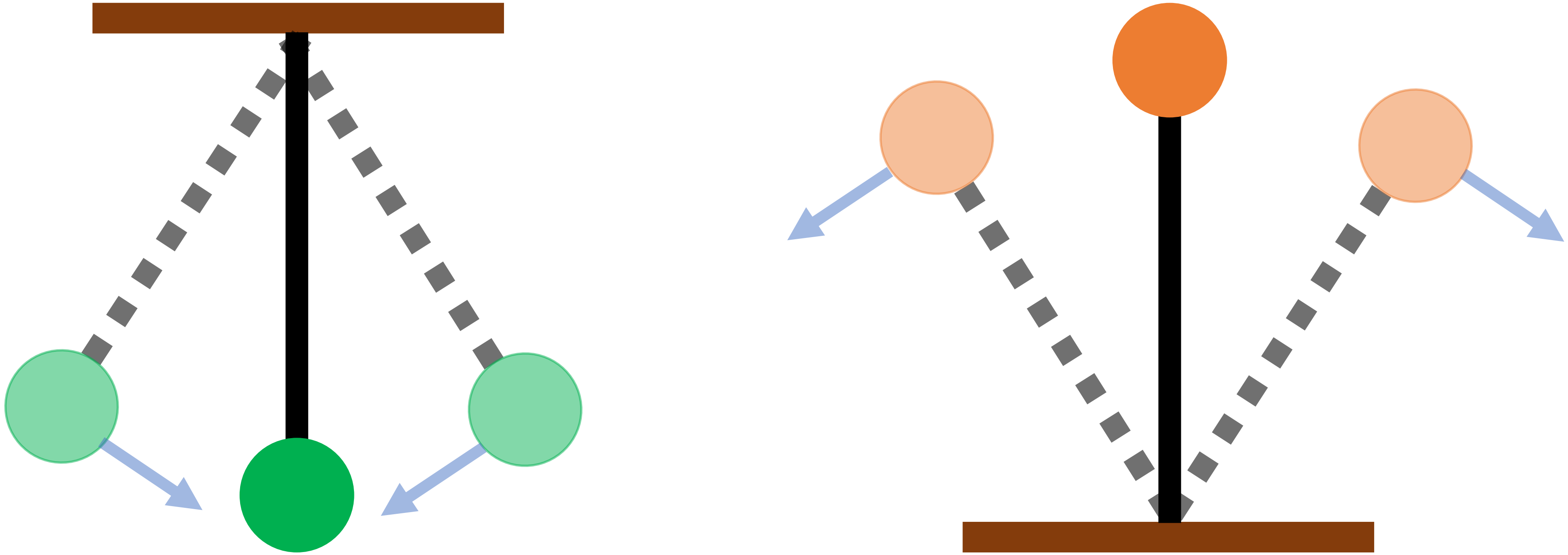
1. Movements, equilibrium
2. **Stability**
3. LTI systems: movements, equilibrium, stability
4. Linearization

Stability



- Both balls are in an **equilibrium.**
- The **green ball** is in a **stable equilibrium.**
- The **orange ball** is in a **unstable equilibrium**

Stability



- The **green ball** is in a **stable equilibrium.**
- The **orange ball** is in an **unstable equilibrium.**

Stability

- If an equilibrium is stable then if there is a small **perturbation on the initial condition** then the system tends to reach the equilibrium.

- Stability is a **property of the equilibrium point** and not of the system.

- The same system can have stable equilibrium and unstable ones (see the previous slide).

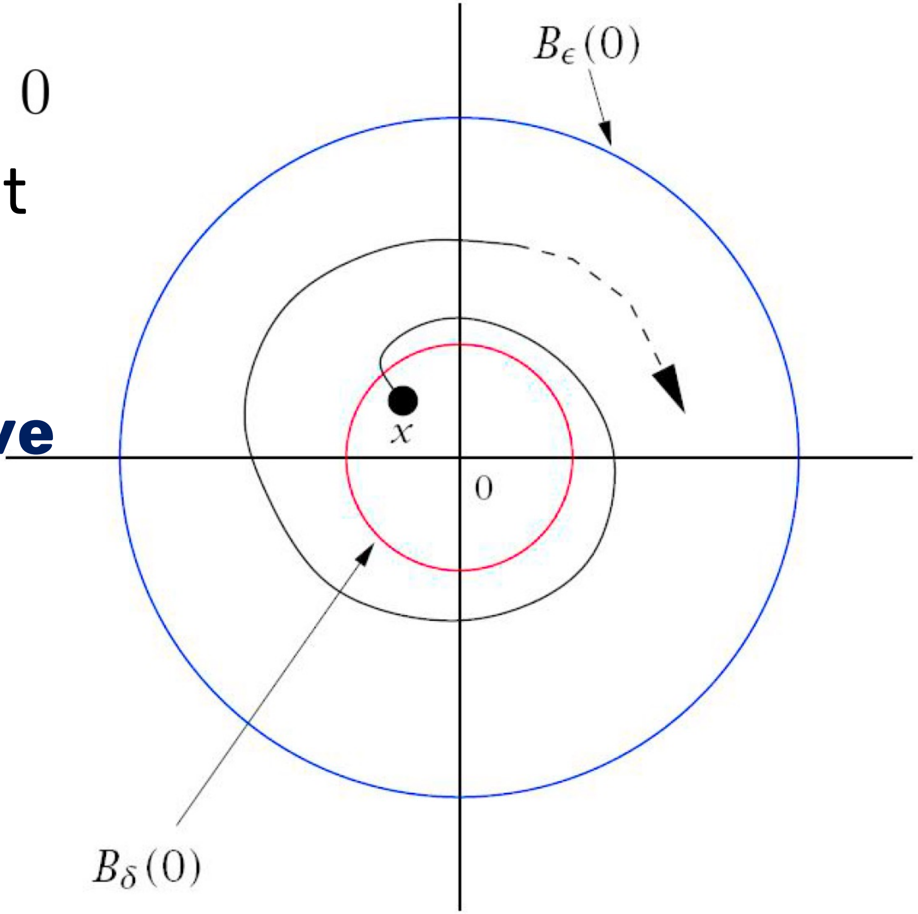
- Stability is a local property of the equilibrium and it works for **small perturbations.**

Stability: formal definition

Stability: The equilibrium point *x=0* is **locally stable** if

$$\forall \epsilon \geq 0 \, \exists \delta \geq 0 \text{ s. t. } \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \, \forall t \geq 0$$

- The equilibrium point *x=0* is **unstable** if it is not **stable.**



Attractivity: The equilibrium point *x=0* is **attractive** if:

$$\lim_{t \to \infty} x(t) = 0$$

Asymptotic Stability: The equilibrium point *x=0* is **asymptotically stable** if it is

Locally Stable + Attractive

Check if an equilibrium is stable

- In general, not an easy problem to solve.

- For linear time-invariant (LTI) system, the solution is quite simple.
  - You just need to check the eigenvalues of matrix A.

- For a nonlinear time-invariant system, one can **linearize** it about the equilibrium point and check stability of the equilibrium using the method for LTI systems.

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LTI Systems

- LTI stands for **Linear Time-Invariant Systems.**
- They are a very specific class of system.
- They are very **simple** to study and there is a lot of theory about them.
- In a first approximation, they can explain a large number of phenomena/processes.

LTI Systems

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \quad x(0) = x_0 \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

SISO

In LTI systems, functions $f(x,u)$ and $g(x,u)$ are linear functions of the form

$$\begin{aligned} x_1(t+1) &= a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) + b_1u(t) \\ x_2(t+1) &= a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) + b_2u(t) \\ &\vdots \\ x_n(t+1) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) + b_nu(t) \\ y(t) &= c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t) + du(t) \end{aligned}$$

LTI Systems

The LTI systems can be rewritten in compact form

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^{n \times 1}
$$

$$
C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \in \mathbb{R}^{1 \times n}, \qquad D = d \in \mathbb{R}
$$

Example

- The LTI systems

$$
\begin{aligned}
x_1(t+1) &= x_1(t) + x_2(t) + 0.5u(t) \\
x_2(t+1) &= x_2(t) + u(t) \\
y(t) &= x_1(t)
\end{aligned}
$$

- SISO
- Order 2

can be rewritten in compact form with

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \in \mathbb{R}^{2\times 1}
$$

$$
C = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2}, \qquad D = 0 \in \mathbb{R}
$$

Movements

The movements of a discrete-time LTI systems can be computed iteratively.

Given $\boldsymbol{u}(t)\ \forall t \geq 0$ and $x(0)$

$$
\begin{aligned}
x(1) &= Ax(0) + Bu(0) \\
x(2) &= Ax(1) + Bu(1) \\
&= A^2x(0) + ABu(0) + Bu(1) \\
x(3) &= Ax(2) + Bu(2) \\
&= A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) \\
&\vdots \\
x(t) &= \boxed{A^t x(0)} + \boxed{\sum_{j=0}^{t-1} A^j Bu(t-j-1)}
\end{aligned}
$$

Forced movement

Free movement

Movements

$$x(t) = \boxed{A^t x(0)} + \boxed{\sum_{j=0}^{t-1} A^j B u(t-j-1)}$$

Free movement

Forced movement

- ➢The **free movement** only depends on the initial condition

- ➢The **forced movement** is forced by the input applied to the system.

Output Movement

It is easy to see that

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= C\left(A^t x(0) + \sum_{j=0}^{t-1} A^j B u(t-j-1)\right) + Du(t) \\
&= \boxed{CA^t x(0)} + \boxed{\sum_{j=0}^{t-1} CA^j B u(t-j-1) + Du(t)}
\end{aligned}
$$

Free movement

Forced movement

Superposition principle

Since LTI systems are linear systems, they enjoy the **superposition principle.**

➢Given two initial condition $x_1(0)$ and $x_2(0)$, and given

$$\boxed{x(0) = \alpha x_1(0) + \beta x_2(0)}$$

then

$$\begin{aligned} x(t) &= A^t x(0) \\ &= A^t \left(\alpha x_1(0) + \beta x_2(0)\right) \\ &= \alpha A^t x_1(0) + \beta A^t x_2(0) \\ &= \boxed{\alpha x_1(t) + \beta x_2(t)} \end{aligned}$$

Free movement
$u(0) = 0$

Superposition principle

- Similarly, given two control sequences

$$\mathbf{u}_1 = \{u_1(0), u_1(0), \ldots, u_1(k)\}$$
$$\mathbf{u}_2 = \{u_2(0), u_2(0), \ldots, u_2(k)\}$$

and given $\mathbf{u} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2$, then

$$\begin{aligned} x(t) &= \sum_{j=0}^{t-1} A^j B u(t-j-1)) \\ &= \sum_{j=0}^{t-1} A^j B \left(\alpha u_1(t-j-1) + \beta u_2(t-j-1)\right) \\ &= \sum_{j=0}^{t-1} A^j B \alpha u_1(t-j-1) + \sum_{j=0}^{t-1} A^j B \beta u_2(t-j-1) \\ &= \boxed{\alpha x_1(t) + \beta x_2(t)} \end{aligned}$$

Forced movement
$x(0) = 0$

Superposition principle

- Combining free and forced movement:

$$
\begin{aligned}
x(t) &= \alpha x_1(t) + \beta x_2(t) \\
&= \alpha A^t x_1(0) + \alpha \sum_{j=0}^{t-1} A^j B u_1(t-j-1) \\
&\quad + \beta A^t x_2(0) + \beta \sum_{j=0}^{t-1} A^j B u_2(t-j-1)
\end{aligned}
$$

Same reasoning holds for the output movements

Equilibrium

Consider the LTI system:

$$\boxed{\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

- Equilibrium: **constant** solution to the difference equation.

$$\boxed{\begin{aligned} \bar{x} &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} + D\bar{u} \end{aligned}}$$

The equilibrium is given by the solution to the previous linear system (first eq. actually).

Equilibrium

Let's do the calculations:

$$\begin{aligned} \bar{x} &= A\bar{x} + B\bar{u} \\ \bar{x} - A\bar{x} &= B\bar{u} \\ (I_n - A)\bar{x} &= B\bar{u} \end{aligned}$$

If $\det(I_n - A) \neq 0$, then

$$\boxed{\bar{x} = (I_n - A)^{-1} B\bar{u}}$$

The equilibrium is univocally defined by the control input:

- ➢ **One equilibrium for each**

If $\det(I_n - A) = 0$, then

The system has infinite solutions or no solution.

Static gain

Consider the case $\det\left(I_n - A\right) \neq 0$

- State equilibrium $\boxed{\bar{x} = (I_n - A)^{-1} B\bar{u}}$

- Output equilibrium:

$$\begin{aligned}\bar{y} &= C\bar{x} + D\bar{u} \\ &= C(I_n - A)^{-1} B\bar{u} + D\bar{u} \\ &= \boxed{\left(C(I_n - A)^{-1} B + D\right)\bar{u}}\end{aligned}$$

- The term $\mu = \left(C(I_n - A)^{-1} B + D\right)$

is called ***static gain*** of the system.

Remarks

- In an LTI system for each value of the input $\bar{u}$ there is a **unique** equilibrium (minor some degenerate cases).

$$\boxed{\bar{x} = (I_n - A)^{-1} B\bar{u}}$$

- The **static gain** allows one to determine how the output changes due to an incremental change in the input, once the system has reached the steady state

$$\boxed{\mu = \left(C(I_n - A)^{-1}B + D\right)} \qquad \Delta\bar{y} = \mu \cdot \Delta\bar{u}$$

Example

$$\begin{cases} x_1(t+1) = 0.5 \cdot x_1(t) + x_2(t) + 3 \cdot u(t) \\ x_2(t+1) = 0.1 \cdot x_2(t) \\ y(t) = x_1(t) + 3 \cdot x_2(t) + 5 \cdot u(t) \end{cases}$$

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \end{bmatrix} \qquad D = 5$$

- Check the determinant:

$$\det(I_n - A) = \det\left(I_n - \begin{bmatrix} 0.5 & 1 \\ 0 & 0.1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0.5 & -1 \\ 0 & 0.9 \end{bmatrix}\right) = 0.45 \neq 0$$

- Compute the static gain:

$$\mu = C \cdot (I_n - A)^{-1} \cdot B + D = \begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & -1 \\ 0 & 0.9 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 5 = 11$$

- Compute the equilibrium with $\bar{u} = 2$ (assuming null initial conditions):

$$\bar{\boldsymbol{x}} = (I_n - A)^{-1} \cdot B \cdot \bar{u} = \begin{bmatrix} 0.5 & -1 \\ 0 & 0.9 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot 2 = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$$

$$\bar{y} = \mu \cdot \bar{u} = 11 \cdot 2 = 22$$

Example

Consider the LTI systems

$$\begin{aligned} x_1(t+1) &= x_1(t) + x_2(t) + 0.5u(t) \\ x_2(t+1) &= x_2(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2}, \qquad D = 0 \in \mathbb{R}$$

➢Check the determinant:

$$\det(I_n - A) = \det\left(I_n - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\right) = 0$$

The system does not have a unique solution.

Stability

Consider the LTI system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

and the equilibrium

$$\begin{aligned} \bar{x} &= (I_n - A)^{-1} B\bar{u} \\ \bar{y} &= \mu\bar{u} \end{aligned}$$

- Is it stable??? Let's check the movements

Stability

- Nominal movement

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(0) &= \bar{x} \end{aligned}$$

$$x(t) = A^t \bar{x} + \sum_{j=0}^{t-1} A^j B \bar{u} = \bar{x}$$

- Perturbated movement

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(0) &= \boxed{\bar{x} + \delta x_0} \end{aligned}$$

$$\begin{aligned} x(t) &= A^t(\bar{x} + \delta x_0) + \sum_{j=0}^{t-1} A^j B \bar{u} \\ &= \boxed{A^t \bar{x} + \sum_{j=0}^{t-1} A^j B \bar{u}} + \boxed{A^t \delta x_0} \\ &= \bar{x} + \underline{A^t \delta x_0} \end{aligned}$$

Stability

$$x(t) = \bar{x} + A^t \delta x_0 \quad \Rightarrow \quad \delta x(t) = A^t \delta x_0$$

The perturbation $\boldsymbol{\delta x}(t)$ corresponds to the free movement with initial condition $x(0) = \boldsymbol{\delta x_0}$.

The perturbation $\boldsymbol{\delta x}(t)$ does not depend on the specific equilibrium.

The entity of the perturbation depends only on the initial perturbation and on the matrix $\boldsymbol{A}$.

Stability

$$x(t) = \bar{x} + A^t \delta x_0 \quad \Rightarrow \quad \delta x(t) = A^t \delta x_0$$

Since the stability depends only on the behavior of the perturbation $\boldsymbol{\delta x}(t)$ and since the perturbation does not depend on the single equilibrium,

- The stability is a property of the **entire system.**
- The equilibriums of an LTI system are all stable or all unstable.
- We can talk of stable, asymptotically stable or unstable **systems**.

Classification

Based on the previous slide, we have 3 possibilities:

- A LTI system is **asymptotically stable** if

$$\lim_{t \to \infty} A^t \delta x_0 = 0$$

- A LTI system is **stable** if $A^t \delta x_0$ is ***bounded***

- A LTI system is **unstable** if

$$\lim_{t \to \infty} A^t \delta x_0 = \pm\infty$$

Example

1.

$$\begin{aligned} x(t+1) &= 0.1x(t) + 0.2u(t) \\ y(t) &= 3x(t) + 2u(t) \end{aligned}$$

$$\lim_{t\to\infty} A^t \delta x_0 = \lim_{t\to\infty} (0.1)^t \delta x_0 = 0$$

Asymptotically stable

2.

$$\begin{aligned} x(t+1) &= -0.3x(t) + 0.2u(t) \\ y(t) &= 3x(t) + 2u(t) \end{aligned}$$

$$\lim_{t\to\infty} A^t \delta x_0 = \lim_{t\to\infty} (-0.3)^t \delta x_0 = 0$$

Asymptotically stable

Example

3.

$$
\begin{aligned}
x(t+1) &= {\color{red}2}x(t) + 0.2u(t) \\
y(t) &= 3x(t) + 2u(t)
\end{aligned}
$$

$$\lim_{t\to\infty} A^t \delta x_0 = \lim_{t\to\infty} 2^t \delta x_0 = \infty$$

Unstable

4.

$$
\begin{aligned}
x(t+1) &= x(t) + u(t) \\
y(t) &= x(t)
\end{aligned}
$$

$$\lim_{t\to\infty} A^t \delta x_0 = \lim_{t\to\infty} 1^t \delta x_0 = \delta x_0$$

Stable

Bounded

Summing up...

Given a first order (n=1) LTI system

Asymptotically stable → $\lim_{t\to\infty} A^t \delta x_0 = 0$ → $\boxed{|A| < 1}$

Stable → $A^t \delta x_0$ bounded → $\boxed{|A| = 1}$

Unstable → $\lim_{t\to\infty} A^t \delta x_0 = \pm\infty$ → $\boxed{|A| > 1}$

Stability: Properties

1. In an asymptotically stable LTI system **the free movement tends to zero**.

$$x_{free}(t) = \lim_{t\to\infty} A^t x_0 = 0$$

2. In an asymptotically stable LTI system **the asymptotic movement depends only on the input**

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \left(\boxed{A^t x_0} + \sum_{j=0}^{t-1} A^j B\bar{u} \right)$$

Goes to zero

Stability: Properties

3. An asymptotically stable LTI system **tends to reach the equilibrium for every initial condition**.

Consider the equilibrium $(\bar{x}, \bar{u})$, then

$$
\begin{aligned}
\lim_{t\to\infty} x(t) &= \lim_{t\to\infty}\left(A^t x_0 + \sum_{j=0}^{t-1} A^j B\bar{u}\right) \\
&= \lim_{t\to\infty}\left(A^t(x_0 + \bar{x} - \bar{x}) + \sum_{j=0}^{t-1} A^j B\bar{u}\right) \\
&= \boxed{\lim_{t\to\infty} A^t(x_0 - \bar{x})} + \lim_{t\to\infty}\left(A^t\bar{x} + \sum_{j=0}^{t-1} A^j B\bar{u}\right) \\
&= \bar{x}
\end{aligned}
$$

Goes to zero

Stability: Properties

4. In an asymptotically stable LTI system there is **one and only one equilibrium** for each $u(k) = \bar{u}$

Consider two different equilibrium states and their movements

$$x(0) = \bar{x}_1 \longrightarrow x_1(t) = A^t \bar{x}_1 + \sum_{j=0}^{t-1} A^j B \bar{u}$$

$$x(0) = \bar{x}_2 \longrightarrow x_2(t) = A^t \bar{x}_2 + \sum_{j=0}^{t-1} A^j B \bar{u}$$

Applying property 3, these movements necessarily converge to the same equilibrium (since the equilibrium input is the same).

Stability: Properties

5. In an asymptotically stable LTI system **if the input is constant than the output tends to a final value**

By applying Property 3 the system converges to an equilibrium, by property 1 the free movements is constant, then

$$\begin{aligned} \bar{x} &= (I_n - A)^{-1} B \bar{u} \\ \bar{y} &= \mu \bar{u} \end{aligned} \qquad \mu = \left(C(I_n - A)^{-1} B + D\right)$$

6. In an asymptotically stable LTI system **if the input is bounded the output is also bounded**

$$|u(t)| \leq \alpha,\, t \geq 0 \quad \Longrightarrow \quad |y(t)| \leq \beta,\, t \geq 0$$

Stability when $n \geq 1$

In this case we look at the ***eigenvalues*** of the matrix A.

Given a matrix $A \in \mathbb{R}^{n\times n}$ the eigenvalue $\lambda \in \mathbb{C}$ and the eigenvector $\boldsymbol{v} \in \mathbb{C}^{n\times 1}$ are the value and the vector such that:

$$A \cdot \boldsymbol{v} = \lambda \cdot \boldsymbol{v}$$

- There are always n eigenvalues and eigenvectors
- If there is a complex eigenvalue there is always its conjugate (complex eigenvalues come in couple).
- The eigenvalues are the root of the characteristic polynomial:

$$\phi(\lambda) = \det(A - \lambda \cdot I_n)$$

Classification

Recalling the stability definitions:

- A LTI system is **asymptotically stable** if $\lim_{t \to \infty} A^t \delta x_0 = 0$
- A LTI system is **stable** if $A^t \delta x_0$ is ***bounded***
- A LTI system is **unstable** if $\lim_{t \to \infty} A^t \delta x_0 = \pm\infty$

Then…

Asymptotic stability vs Instability

Theorem 1

An LTI system is **asymptotically stable** if and only if all the eigenvalues λ_i of the matrix A have norm strictly smaller than one:

$$\forall i, |\lambda_i| < 1 \Leftrightarrow \text{Asymptotically stable}$$

Theorem 2

An LTI system is **unstable** if there is **at least one** eigenvalues λ_i of the matrix A with norm strictly greater than one:

$$\exists i \; s.t \; |\lambda_i| > 1 \Rightarrow \text{Unstable}$$

Simple stability

Theorem 3

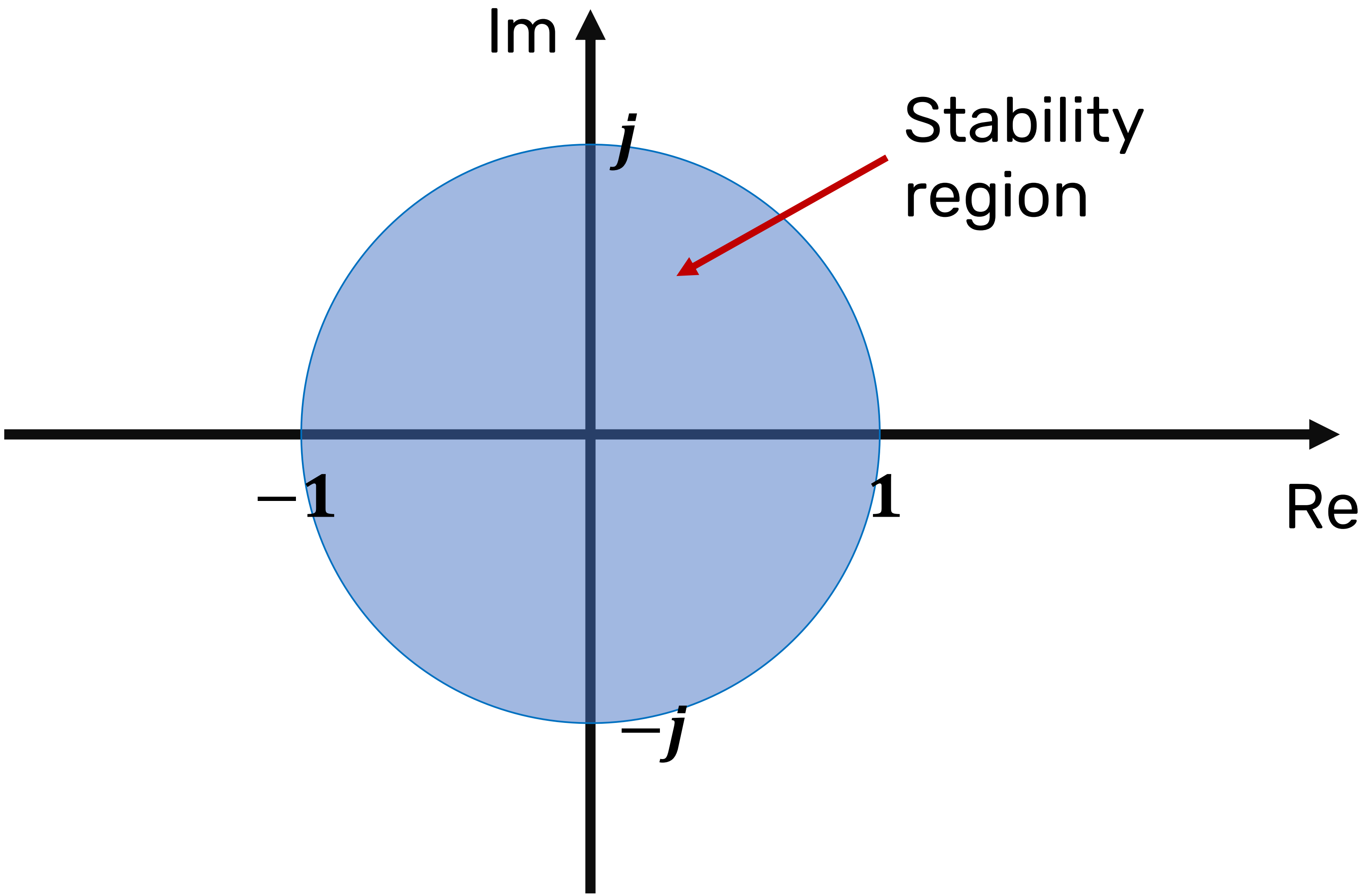
An LTI system is **simply stable** if all the eigenvalues λ_i of the matrix A have norm smaller than one and there is **one and only one** eigenvalue with norm equal to one (or a couple of complex eigenvalues):

$$\forall i, |\lambda_i| \le 1 \; \exists! \, i \; s.t. \, |\lambda_i| = 1 \Rightarrow \text{Simply stable}$$

Remark

1. *A couple of complex eigenvalues counts as one eigenvalue. Therefore, if all the eigenvalues have norm smaller than one except for a couple of complex eigenvalues with norm equal to one the system is simply stable.*
2. *If there are more than one eigenvalues with norm equals to one the system can be unstable or simply stable, more analysis is needed.*

Stability Region



Example

$$A = \begin{bmatrix} 0.1 & 1 \\ 0 & -0.2 \end{bmatrix}$$

$$\begin{aligned} \phi(\lambda) &= \det\left(A - \lambda I_n\right) \\ &= \det \begin{bmatrix} 0.1 - \lambda & 1 \\ 0 & -0.2 - \lambda \end{bmatrix} \\ &= (0.1 - \lambda)(-0.2 - \lambda) \end{aligned}$$

$$\boxed{\begin{aligned} \lambda_1 &= 0.1 \\ \lambda_2 &= -0.2 \end{aligned}}$$

asymptotically stable

Example

$$A = \begin{bmatrix} 1.2 & 0 \\ 1 & 0.9 \end{bmatrix}$$

$$\begin{aligned} \phi(\lambda) &= \det\left(A - \lambda I_n\right) \\ &= \det \begin{bmatrix} 1.2-\lambda & 0 \\ 1 & 0.9-\lambda \end{bmatrix} \\ &= (1.2-\lambda)(0.9-\lambda) \end{aligned}$$

$$\boxed{\begin{aligned} \lambda_1 &= 1.2 \\ \lambda_2 &= 0.9 \end{aligned}}$$

unstable

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\begin{aligned} \phi(\lambda) &= \det(A - \lambda I_n) \\ &= \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 0.5-\lambda \end{bmatrix} \\ &= (1-\lambda)(0.5-\lambda) \end{aligned}$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 0.5 \end{aligned}}$$

Simply stable

Example

Consider the following matrix A of a LTI system

$$A = \begin{bmatrix} 1-\alpha & \beta \\ 0 & 0.1 \end{bmatrix}$$

Determine the values of α and β that make the system stable. The eigenvalues are: $\begin{cases} \lambda_1 = 1-\alpha \\ \lambda_2 = 0.1 \end{cases}$

Therefore, the system is asymptotically stable if and only if:

$$|1-\alpha| < 1 \Rightarrow \begin{cases} 1-\alpha < 1 \Rightarrow \alpha > 0 \\ 1-\alpha > -1 \Rightarrow \alpha < 2 \end{cases} \Rightarrow 0 < \alpha < 2$$

Furthermore, the system is simply stable if:

$$|1-\alpha| = 1 \Rightarrow \begin{cases} 1-\alpha = 1 \Rightarrow \alpha = 0 \\ 1-\alpha = -1 \Rightarrow \alpha = 2 \end{cases} \Rightarrow \alpha = 0 \text{ or } \alpha = 2$$

Outline

1. Movements, equilibrium
2. Stability
3. LTI systems: movements, equilibrium, stability
4. **Linearization**
5. Continuous time systems

What about nonlinear systems?

- We **cannot** talk of stability of a nonlinear systems.
- Recall that stability is a local property, that holds in a neighborhood of an equilibrium point.
- For nonlinear systems, we want to check the stability property of the equilibrium (not the entire system).
- How to do that? We can **linearize** a system in a certain equilibrium and then study the stability of the obtained linearized system using the same tool as for LTI systems.

Linearization

Take a nonlinear model.

$$\boxed{\begin{aligned} x(t+1) &= f(x(t), u(t)), \quad x(0) = x_0 \\ y(t) &= g(x(t), u(t)) \end{aligned}}$$

Let's say $(\bar{x}, \bar{u})$ is an equilibrium, such that $\bar{x} = f(\bar{x}, \bar{u})$

Consider the **Taylor expansion** of $f(x, u)$ around such equilibrium.

$$\underbrace{f(x(t), u(t))}_{x(t+1)} = \underbrace{f(\bar{x}, \bar{u})}_{\bar{x}} + \left.\frac{\partial f(x,u)}{\partial x}\right|_{(\bar{x},\bar{u})} (x(t) - \bar{x}) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{(\bar{x},\bar{u})} (u(t) - \bar{u})$$

Linearization

Define now $\delta x(t) = (x(t) - \bar{x}), \quad \delta u(t) = (u(t) - \bar{u})$

Then

$$\boxed{\delta x(t+1) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(\bar{x},\bar{u})} \delta x(t) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{(\bar{x},\bar{u})} \delta u(t)}$$

$$(x(t+1) - \bar{x})$$

This approximation is linear in $\delta x(t)$ and $\delta u(t)$

- Same reasoning hold for the output transformation

Linearized system

Then we have

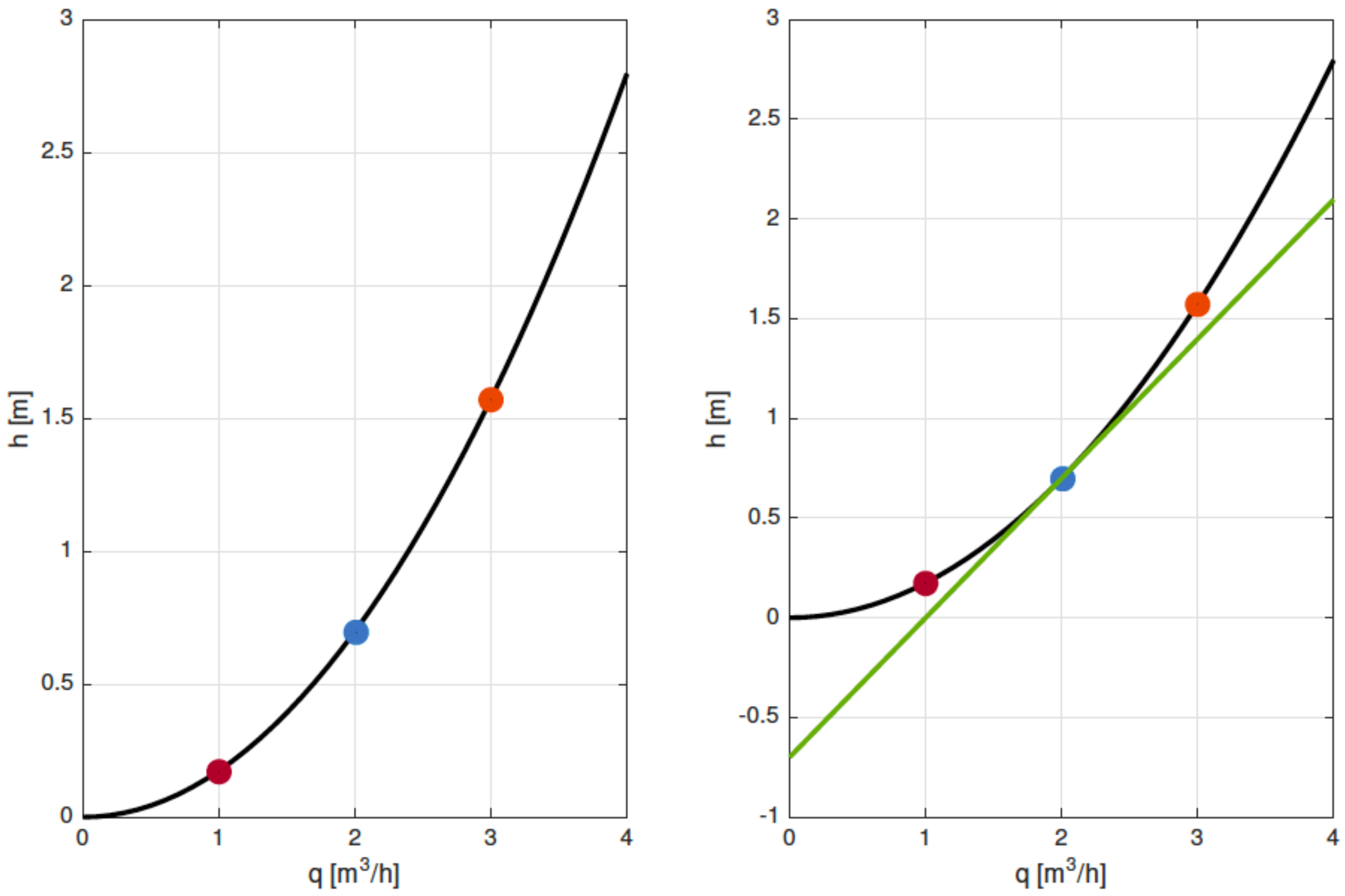
$$\begin{aligned}\delta x(t+1) &= A\delta x(t) + B\delta u(t)\\ \delta y(t) &= C\delta x(t) + D\delta u(t)\end{aligned}$$

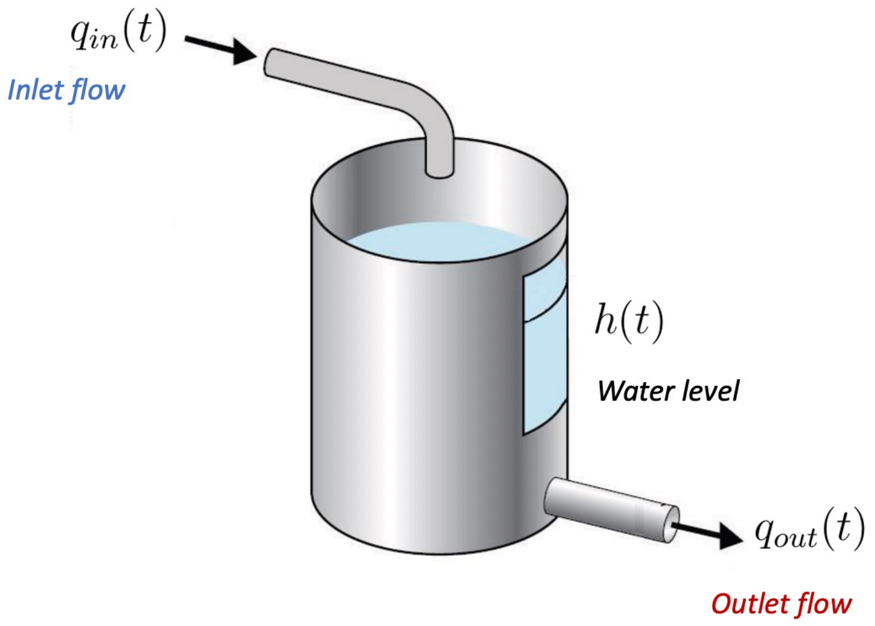
with

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(\bar{x},\bar{u})} \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(\bar{x},\bar{u})} \quad C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{(\bar{x},\bar{u})} \quad D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{(\bar{x},\bar{u})}$$

- Thus, we can study the stability of the equilibrium by analyzing the stability of the **linearized system** using the same tool as for LTI system.
- **Indirect Lyapunov method.** It also holds for continuous time system.

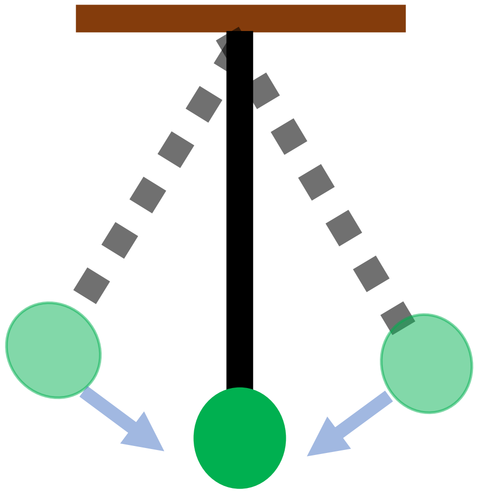
Example: water tank





$$
\begin{aligned}
x(t+1) &= x(t) + \frac{\Delta t}{A}\left(u(t) - \kappa\sqrt{x(t)}\right) \\
y(t) &= \kappa\sqrt{x(t)}
\end{aligned}
$$

Example: Linearized system



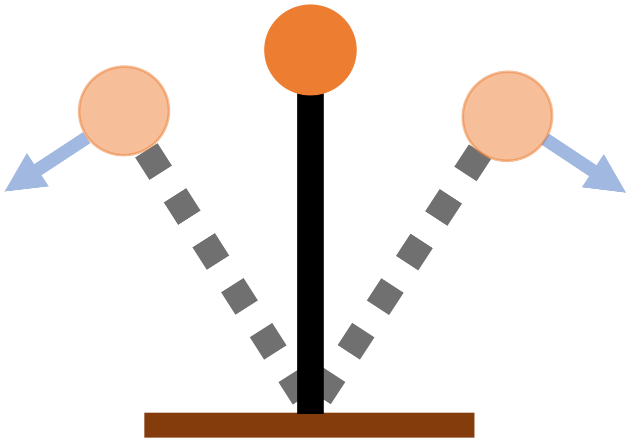
Consider the discrete time of a pendulum

$$
\begin{aligned}
x_1(k+1) &= x_1(k+1) + \Delta_t x_2(k), \\
x_2(k+1) &= x_2(k) + \Delta_t \left(-\frac{g}{l}\sin(x_1(k)) - \frac{k}{m}x_2(k) + \frac{1}{ml^2}u(k) \right)
\end{aligned}
$$

with l=1m, m=1, k=0.5, g=9.81, $\Delta_t = 0.1\ s$.

- This system has two equilibria for u(k)=0

$$
\begin{aligned}
\bar{x}_a &= (0,0) \\
\bar{x}_b &= (\pi,0)
\end{aligned}
$$



Example: Linearized system

We can study the stability of these two equilibria, by linearizing about such points

$$\begin{aligned}\delta x_1(k+1) &= \delta x_1(k) + 0.01\delta x_2(k),\\ \delta x_2(k+1) &= -0.01g\cos(\bar{x}_1)\delta x_1(k) + (1-0.01k)\delta x_2(k) + 0.01\delta u(k)\end{aligned}$$

Then we can write down matrices A and B, as:

$$A = \begin{bmatrix} 1 & 0.01 \\ -0.0981\cos(\bar{x}_1) & (1-0.005) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

Which should be evaluated in the two equilibria
$$\begin{aligned}\bar{x}_a &= (0,0)\\ \bar{x}_b &= (\pi,0)\end{aligned}$$

Example: Linearized system

Let's consider the first equilibrium

$$\bar{x}_a = (0,0), \quad A = \begin{bmatrix} 1 & 0.01 \\ -0.0981 & 0.995 \end{bmatrix}$$

Whose eigenvalues are

$$\begin{aligned} \lambda_1 &= 0.9975 + j0.0312 \\ \lambda_2 &= 0.9975 + j0.0312 \end{aligned}$$

Since both eigenvalues are such that

$$|\lambda_i| = 0.9980 < 1$$

Then this equilibrium is **asymptotically stable.**

Example: Linearized system

Let's consider the second equilibrium

$$\bar{x}_b = (\pi, 0), \quad A = \begin{bmatrix} 1 & 0.01 \\ 0.0981 & 0.995 \end{bmatrix}$$

Whose eigenvalues are

$$\begin{aligned} \lambda_1 &= 1.0289 \\ \lambda_2 &= 0.9661 \end{aligned}$$

Since $|\lambda_1| > 1$ then this equilibrium is **unstable**.

Outline

1. Movements, equilibrium
2. Stability
3. LTI systems: movements, equilibrium, stability
4. Linearization
5. **Continuous time systems**

State-Space Representation

The generic state-space representation of a time-invariant nonlinear dynamical system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(\boldsymbol{t}), u(t)) \\ y(t) = g\big(\boldsymbol{x}(\boldsymbol{t}), u(t)\big) \end{cases}$$ → **State Equation** / **Output Equation**

$\boldsymbol{x}(t) \in \Re^{\boldsymbol{n}}$

$$\boldsymbol{x}(t_0) = \boldsymbol{x_0}$$ → **Initial state**

State variables are internal variables ($\boldsymbol{x}(\boldsymbol{t})$) of the system whose knowledge at the time t_0 is the minimum amount of information needed to determine the output $\boldsymbol{y}(\boldsymbol{t})$ due to the **input** $\boldsymbol{u}(\boldsymbol{t})$, far all $t > t_0$

SISO → Single Input Single Output

$u(t) \in \Re$ scalar

$y(t) \in \Re$ scalar

MIMO → Multi Input Multi Output

$u(t) \in \Re^{\boldsymbol{m}}$ array

$y(t) \in \Re^{\boldsymbol{p}}$ array

State-Space Representation

When there are no input variables, the system

$$\dot{\boldsymbol{x}}(t) = f\big(\boldsymbol{x}(\boldsymbol{t})\big)$$

Is defined as **autonomous**.

When the function $f(\boldsymbol{x}, \boldsymbol{u})$ is linear in $\boldsymbol{x}(\boldsymbol{t})$ e $\boldsymbol{u}(\boldsymbol{t})$, the system is **linear time-invariant** (LTI):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Con $\boldsymbol{A} \in \Re^{\boldsymbol{n},\boldsymbol{n}}$, $\boldsymbol{B} \in \Re^{\boldsymbol{n},\boldsymbol{m}}$, $\boldsymbol{C} \in \Re^{\boldsymbol{p},\boldsymbol{n}}$ e $\boldsymbol{D} \in \Re^{\boldsymbol{p},\boldsymbol{m}}$.

Equilibrium

If we enter constant inputs $u(t) = \bar{u}$ We obtain movements of the state and output that are also constant over time.

These movements are called **equilibrium states and outputs**. Equilibrium states must satisfy the equation $\dot{x}(t) = 0$

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = g(\boldsymbol{x}(t), u(t)) \end{cases}$$

$$u(t) = \bar{u}\,, t \geq t_0$$

$$\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = \boldsymbol{0}$$

State of Equilibrium

Movement of the states $x(t) = \bar{x}$ constant over time with

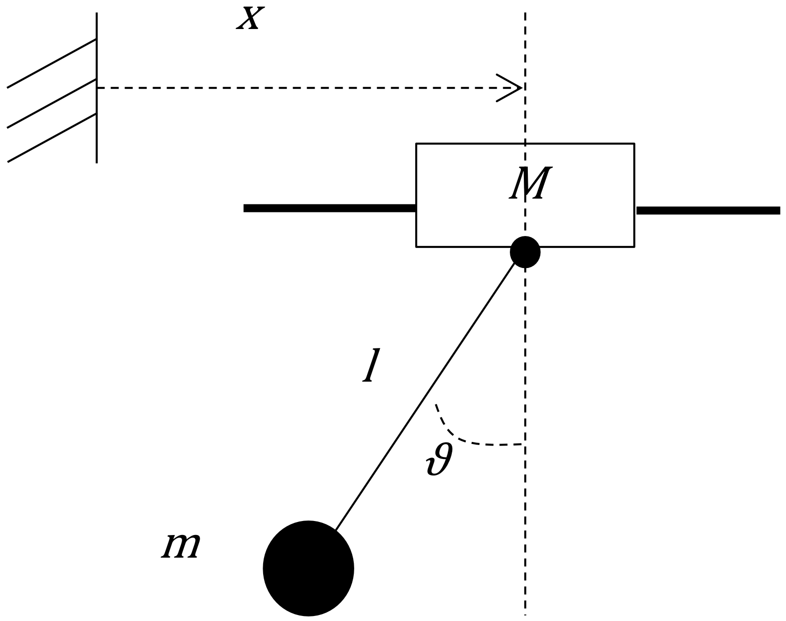
$u(t) = \bar{u}$

Equilibrium output

Movement of the output $y(t) = \bar{y}$ constant over time with

$u(t) = \bar{u}$

Example



$$\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = -\left(\frac{u(t)}{l} \cos x_1(t) + \frac{g}{l} \sin x_1(t) + \frac{b}{ml^2} x_2(t)\right) \\ y(t) = x_1(t) \end{cases}$$

$$\boldsymbol{x(t)} = \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\dot{\theta}} \end{bmatrix} \qquad \boldsymbol{f(\bar{x}, \bar{u}) = 0} \qquad \boldsymbol{\bar{u} = 0}$$

$$\begin{cases} 0 = \bar{x}_2 \\ 0 = -\left(\frac{\bar{u}}{l} \cos \bar{x}_1 + \frac{g}{l} \sin \bar{x}_1 + \frac{b}{ml^2} \bar{x}_2\right) \\ \bar{y} = \bar{x}_1 \end{cases}$$

$$\begin{cases} \bar{x}_2 = 0 \\ 0 = -\left(\frac{g}{l} \sin \bar{x}_1\right) \\ \bar{y} = \bar{x}_1 \end{cases}$$

Equilibria:

$$\boldsymbol{\bar{x}} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$$

Equilibrium of LTI systems

Let's assess the presence of equilibrium in LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Let's say $\dot{x}(t) = 0$ at $u(t) = \bar{u}$

$$0 = A\bar{x} + B\bar{u} \Longrightarrow A\bar{x} = -B\bar{u} \Longrightarrow \bar{x} = -A^{-1}B\bar{u}$$

$\boxed{\det(A) \neq 0}$

The equilibria are: $A\bar{x} = -B\bar{u}$

$\boxed{\det(A) = 0}$

The system $A\bar{x} = -B\bar{u}$ can have

- infinite solutions
- No solution

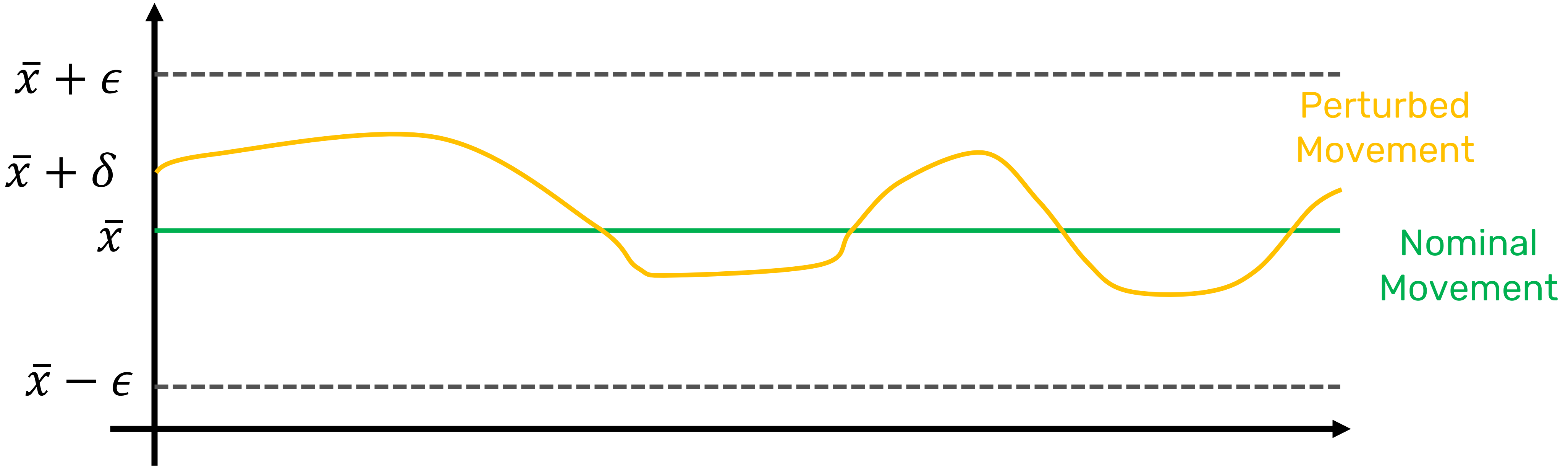
Stability

An equlibrium $\bar{\mathbf{x}}$ is said to be stable if, for each $\epsilon > 0$ there esists $\delta > 0$ such that for each initial state x_0 that satisfies:

$$\|x_0 - \bar{x}\| \leq \delta$$

It results

$$\|x(t) - \bar{x}\| \leq \epsilon \quad t \geq 0$$



Stability

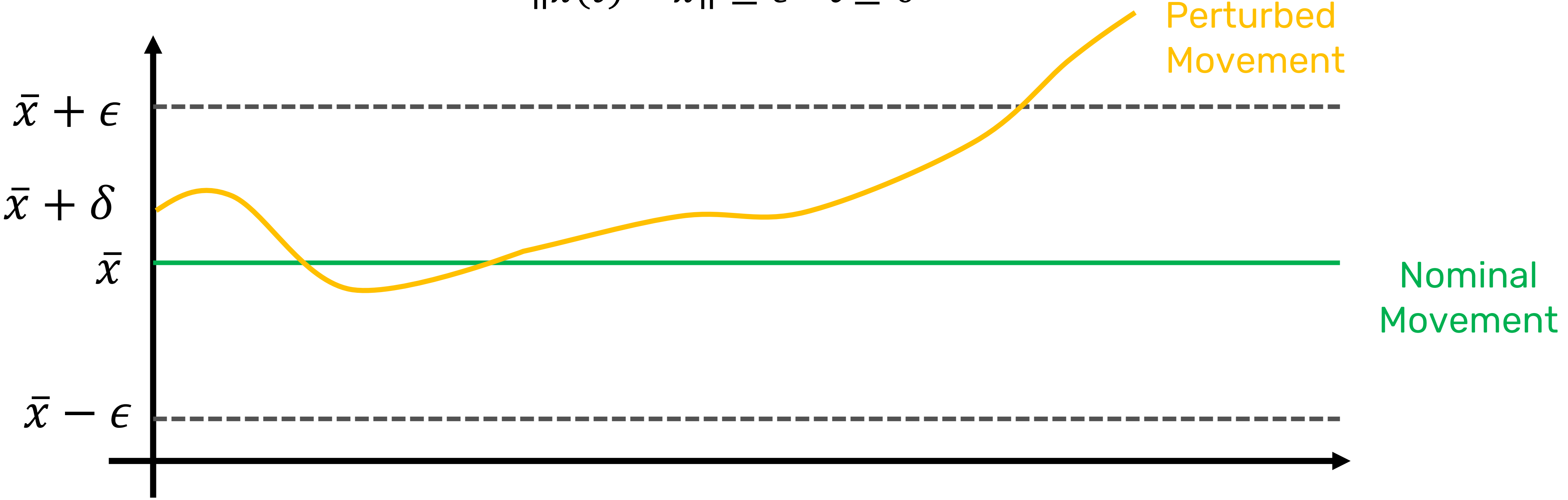
An equilibrium $\bar{\mathbf{x}}$ It is said to be **unstable** if it is not stable.

For each$\epsilon > 0$ **does not exist** $\delta > 0$ such that for each initial state x_0 that satisfies:

$$\|x_0 - \bar{x}\| \leq \delta$$

It results

$$\|x(t) - \bar{x}\| \leq \epsilon \quad t \geq 0$$



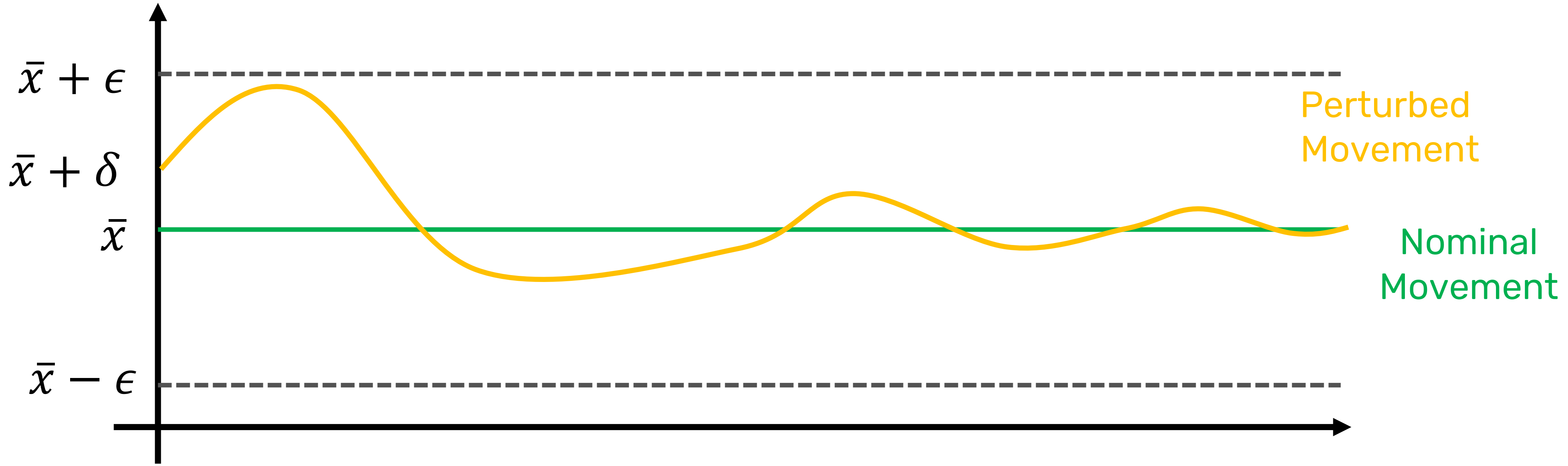
Stability

An equilibrium $\bar{\mathbf{x}}$ is said to be asymptotically stable if, for each $\epsilon > 0$ Exists $\delta > 0$ such that for all initial states x_0 that satisfy:

$$\|x_0 - \bar{x}\| \leq \delta$$

It results

$$\|x(t) - \bar{x}\| \leq \epsilon \quad t \geq 0 \quad e \quad \lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$$



Stability of LTI systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The nominal movement of an LTI system is given by Lagrange's formula:

$$x(t) = e^{At}x_{t0} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Assuming a perturbation of the initial condition $x_{t0} = \bar{x} + \delta_{\bar{x}}$ We get the perturbed movement:

$$\tilde{x}(t) = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + e^{At}\delta_{\bar{x}}$$

Stability of LTI systems

$$\tilde{x}(t) = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + e^{At}\delta_{\bar{x}}$$

The perturbed movement differs from the nominal movement only in that $\delta x(t) = e^{At}\delta_{\bar{x}}$. We can therefore deduce that, for an LTI system:

- The perturbed movement does not depend on the particular state of equilibrium. We can therefore speak of the stability of the system (→ **global property**)
- The difference between the nominal and the perturbed movement depends on the values assumed by the matrix A

Stability of LTI systems

$$\tilde{x}(t) - \bar{x} = e^{At}\delta_{\bar{x}}$$

We can deduce that:

- **Asymptotically stable system** $\lim_{t\to\infty} e^{At} = 0$
- **Unstable system** e^{At} diverges with $t \to \infty$
- **Stable System** e^{At} bounded $\forall t$

Stability theorem of LTI systems

1. A (continous time) LTI system is **asymptotically stable** if and only if all eigenvalues of matrix A have **negative real part**

$$Re(s_i) < 0, \qquad \forall i$$

2. An LTI system is **unstable** if matrix A has at least **one eigenvalue with positive real part**

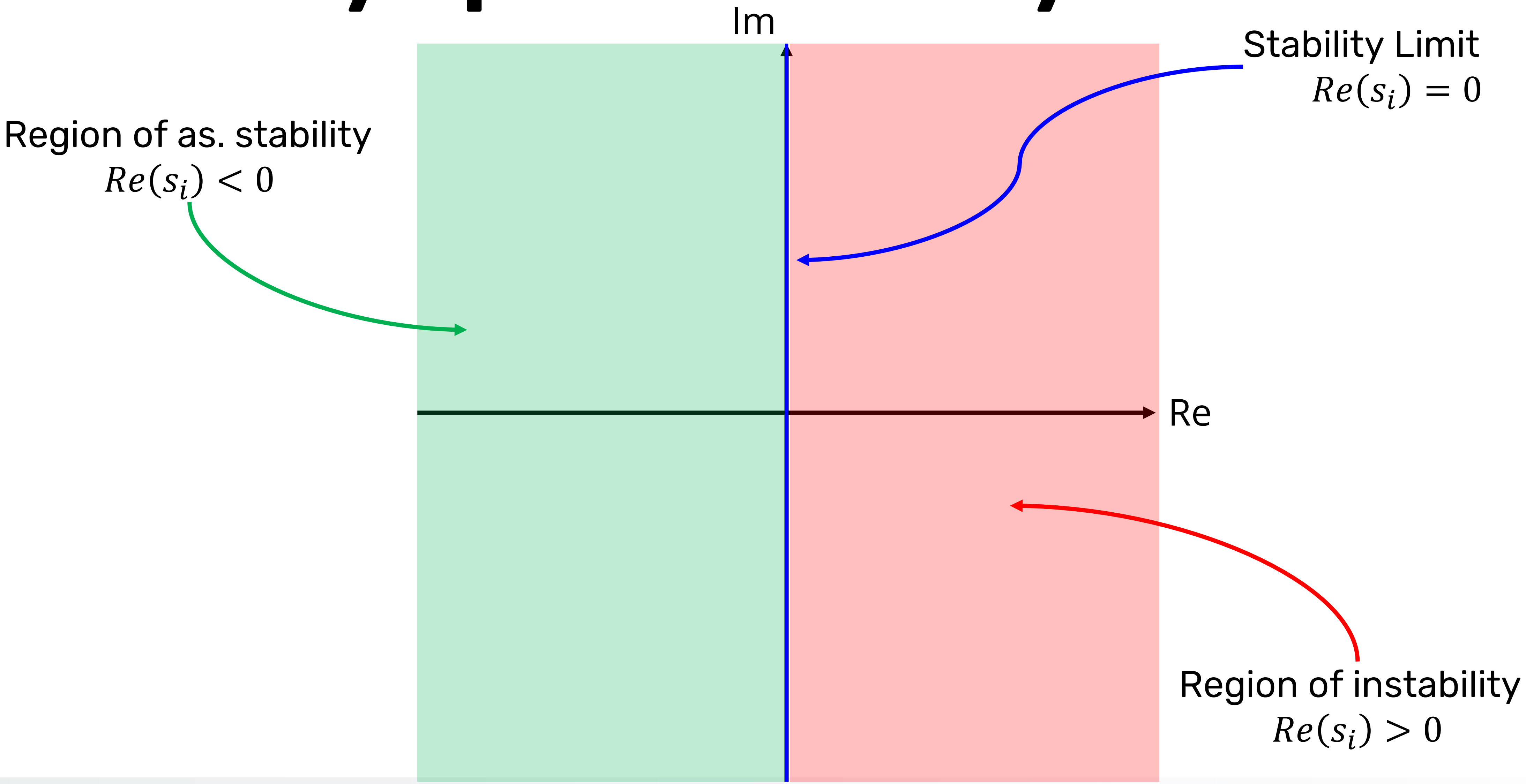
$$\exists i^*: Re\left(s_{i^*}\right) > 0$$

3. An LTI system is **stable** if matrix A has **all eigenvalues with negative real part and one null**

$$Re(s_i) < 0, \qquad \forall i$$

$$\exists!\, i^* : Re(s_{i^*}) = 0$$

Area of asymptotic stability



Properties of LTI systems

1. An as. Stable LTI system, if perturbed, tends to return to equilibrium before the perturbation.
2. At any constant input $\bar{u}$ is associated **one and only one** state of equilibrium $\bar{x}$
3. **A system as. stable is not affected by the initial conditions** (the movement of the state depends only on $u(t)$)
4. With zero input, the movement of the state tends asymptotically to zero.
5. With $u(t) = \bar{u}$ the output of an as. stable system tends to the stationary value $\bar{y}$.
6. **If the input is bounded, the output of an as. Stable LTI system will also be bounded**

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