UNIVERSITÀ
DEGLI STUDI
DI BERGAMO

Dipartimento
di Ingegneria Gestionale,
dell'Informazione e della Produzione

Lesson 2.

Dynamical Systems: Introduction and classification

CONOTRL AND MODELING OF BIOLOGICAL SYSTEMS

MASTER DEGREE IN MEDICAL ENGINEERING

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Outline

1. Introduction to dynamical systems

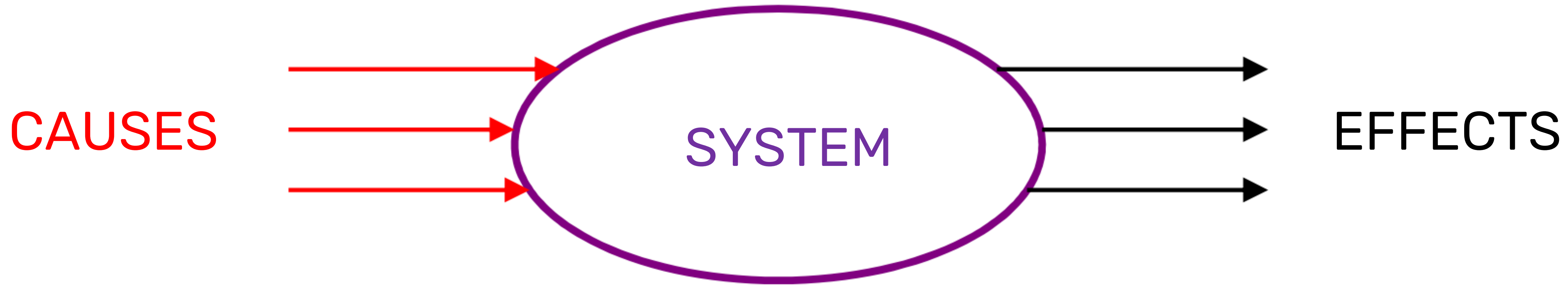
2. Classification of dynamical systems

Outline

1. **Introduction to dynamical systems**

2. Classification of dynamical systems

Dynamic Systems

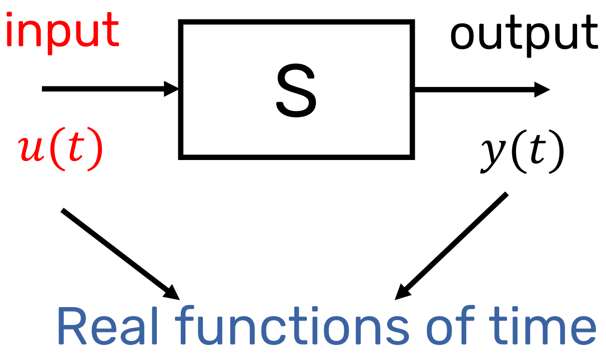


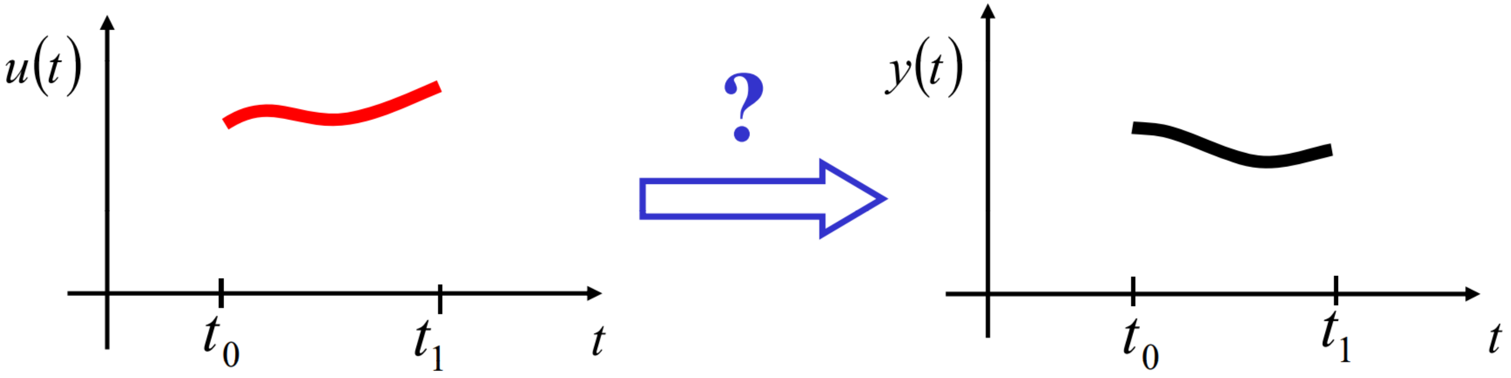
A dynamic system is an agent that interacts with the surrounding world, by means of some **causes (inputs)** operating on it and which determines some **effects (outputs)**, that are the answer of the system to such stimulations.

We need mathematical models to describe systems.

Dynamic Systems

What does **dynamic** means?





The knowledge of the input at time t **is not enough** to univocally determine the value of the output at time t.

We need some kind of memory. What kind of mathematical relation can express it?

What is a mathematical model?

- **Mathematical models** are mathematical objects that can be used to describe, analyze, simulate the behavior of a dynamic system.

- **Dynamic systems**: phenomena or physical systems whose properties change with time.
  - The spread of an infectious disease: phenomenon.
  - Dynamic of an airplane: physical system.

- A mathematical model is a **set of equations** that explains the relation between the variable involved in the phenomenon/system.

- They represent only a simplified version of the real phenomena.

Mathematical Models Classification

- **Continuous-time model**: set of **differential equations** that describes the dynamical behavior of a phenomenon/system over time $t \in \mathbb{R}$

- **Discrete-time model**: set of **difference equations** that describes the dynamical behavior of a phenomenon/system over time $\mathrm{t} \in \mathbb{Z}$

- **Static model**: simple static equation that describes the behavior of a phenomenon/system without considering the relation between time instants (i.e. Ohm's Law: $v = R * i$)

Example 1: SIR model

- Mathematical models describing the spread of an Infectious Disease (*Kermack and McKendrick, 1927*)

- S(t): susceptible subjects (not infected) at time *t*
- I(t): infected subjects at time *t*
- R(t): removed subjects at time *t* (either recovered or dead, cannot be infected again)
- N: constant number of subjects in the population

$$N = S(t) + I(t) + R(t),\ \forall t \geq 0$$

Example 1: SIR model

Infections rate

$$\dot{S}(t) = -\frac{\beta S(t) I(t)}{N}$$

Recovered/death rate

$$\dot{I}(t) = \frac{\beta S(t) I(t)}{N} - \gamma I(t)$$

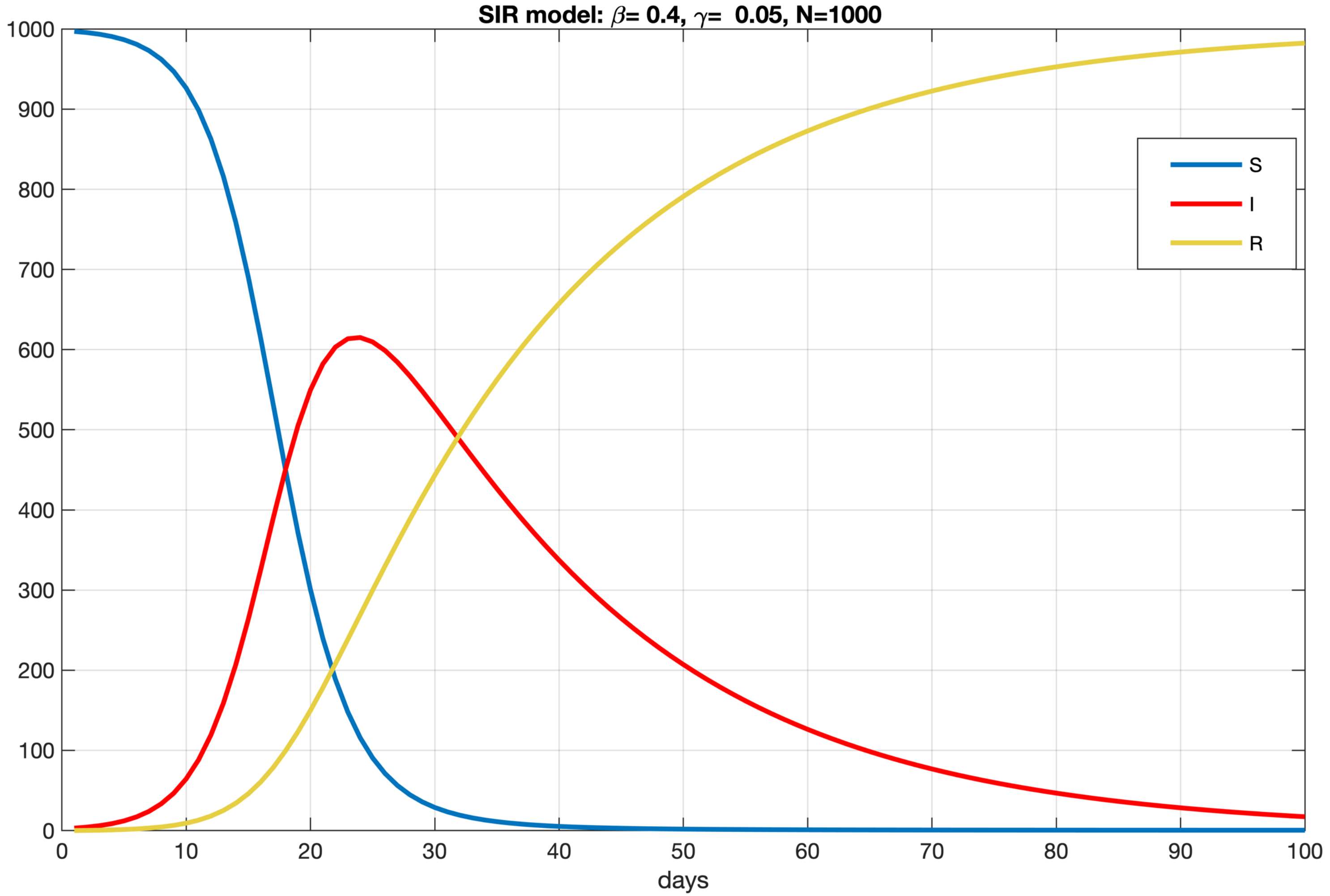
$$\dot{R}(t) = \gamma I(t)$$

$$\dot{S} = \frac{dS}{dt}, \quad \dot{I} = \frac{dI}{dt}, \quad \dot{R} = \frac{dR}{dt}$$

$$R_0 = \frac{\beta}{\gamma}$$

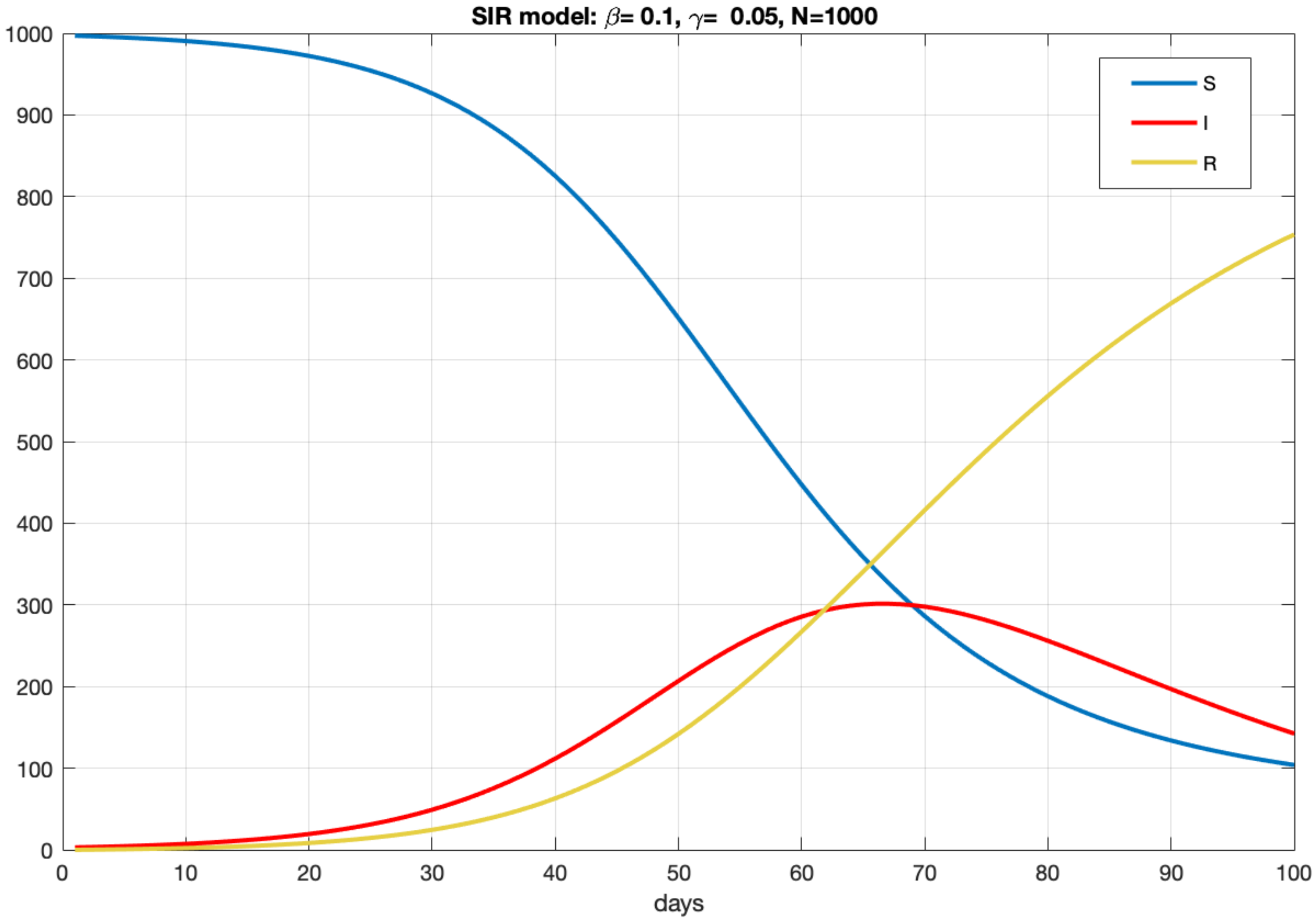
Basic reproduction number: the spread of an infectious disease starts if >1

Example 1: SIR model



$$\beta = 0.4, \quad \gamma = 0.05$$

Example 1: SIR model



$$\beta = 0.1, \quad \gamma = 0.05$$

Flattening the curve with social distancing…

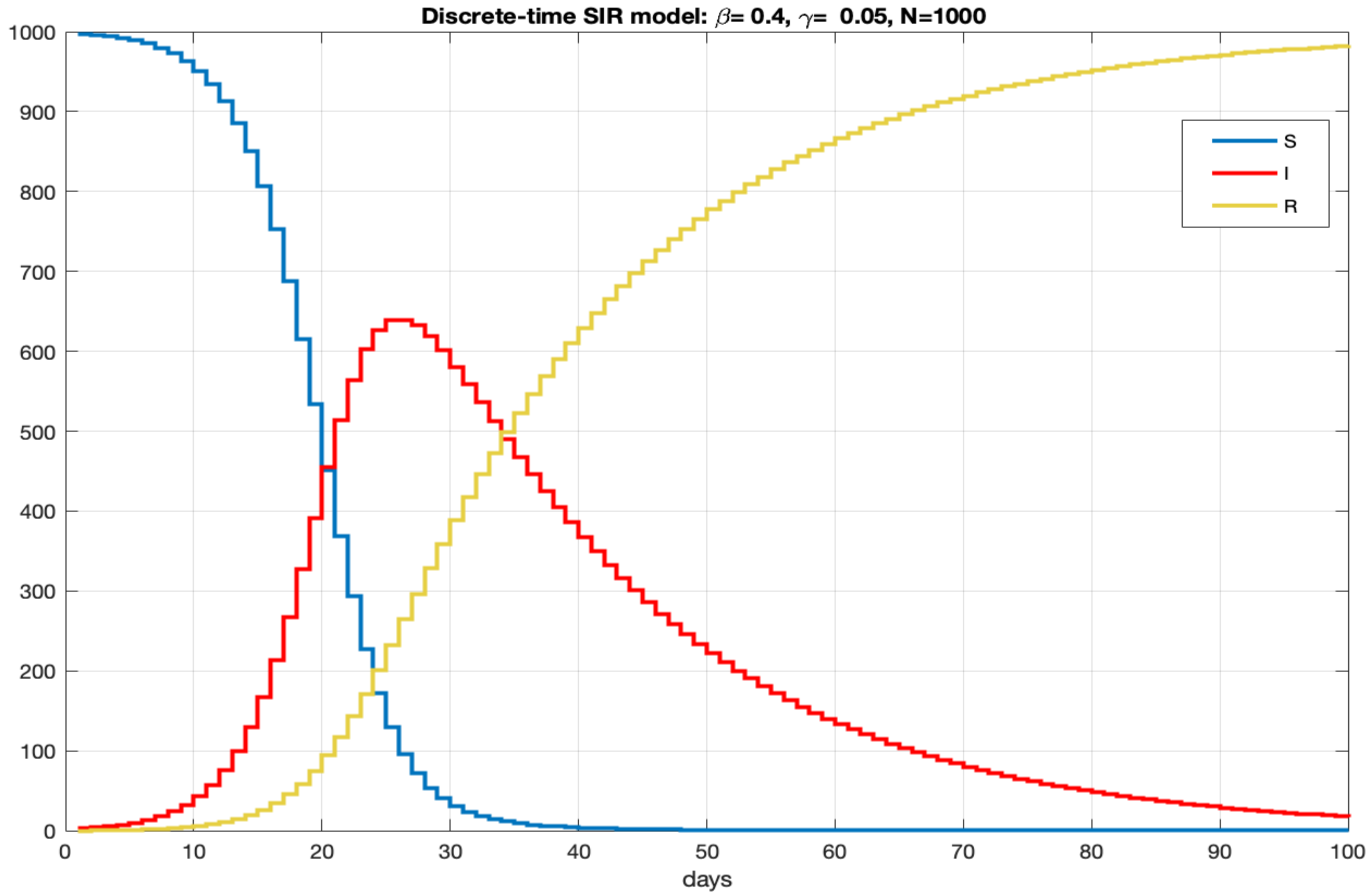
Example 2: SIR model in discrete time

Difference equations

$$
\begin{aligned}
S(t+1) &= S(t) - \frac{\beta S(t) I(t)}{N} \\
I(t+1) &= I(t) + \frac{\beta S(t) I(t)}{N} - \gamma I(t) \\
R(t+1) &= R(t) + \gamma I(t)
\end{aligned}
$$

Future values of S, I, R depend on the past

Example 2: SIR model in discrete time



t is the time step

In this case t =1 day

Example 2: SIR model in discrete time

- SIR models are a simplified mathematical generalization of a certain phenomenon (seasonal flu, COVID, Ebola outbreaks, etc.).

- A lot of assumptions are made in order to make such a model work
  - *N is constant*
  - *Every infected subject needs the same time to recover or die.*

- Without assumption, models would be more complex (i.e. a flight simulators, F1 simulators, weather forecast models, etc.).

- A complex model is not always the best choice.

"All models are wrong, but some are useful"

Input and Output variables

- SIR models are **autonomous systems**: their variables evolve "by themselves", there is no external action on the system.

- In general, dynamic systems are subject to the action of external signals (that do not depend on the dynamic of the system), which manipulate their behavior.

  - **Controlled Input**: external signals that one can manipulate to make a system behave as one desire. They can be manipulated
  - **Disturbances**: unwanted external signals that one cannot manipulate. They can either be eliminated or not.

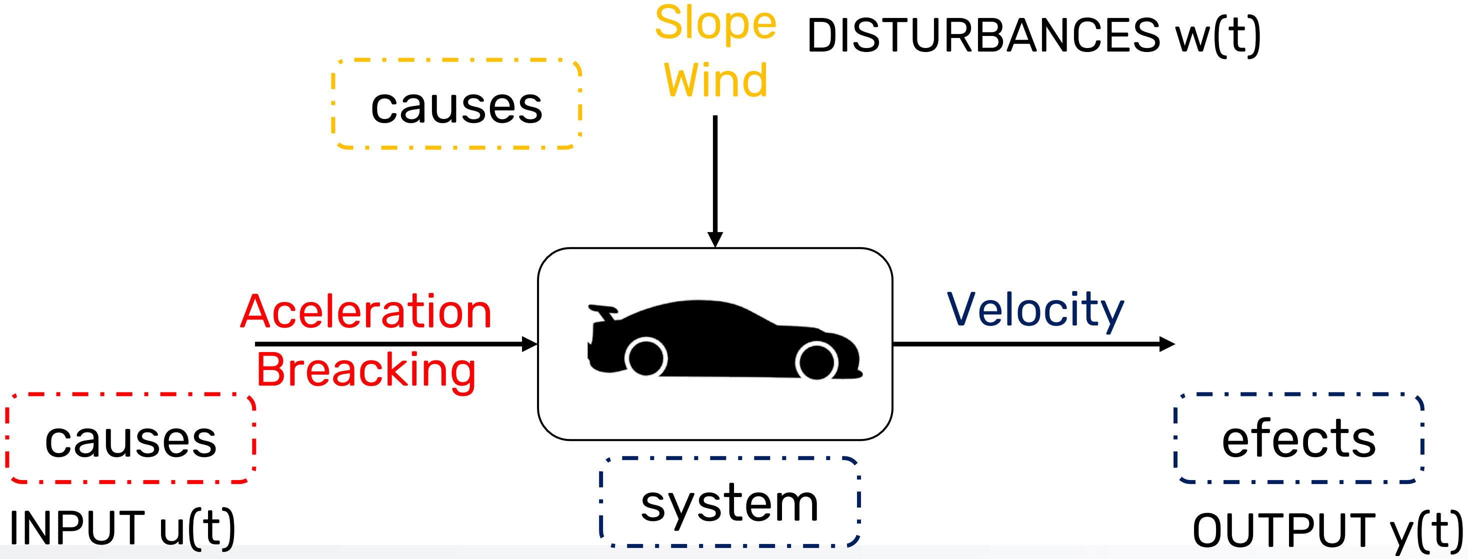
- We can also define the **output** of a system as the outcome of the phenomenon that can be measured.

Take a shower....

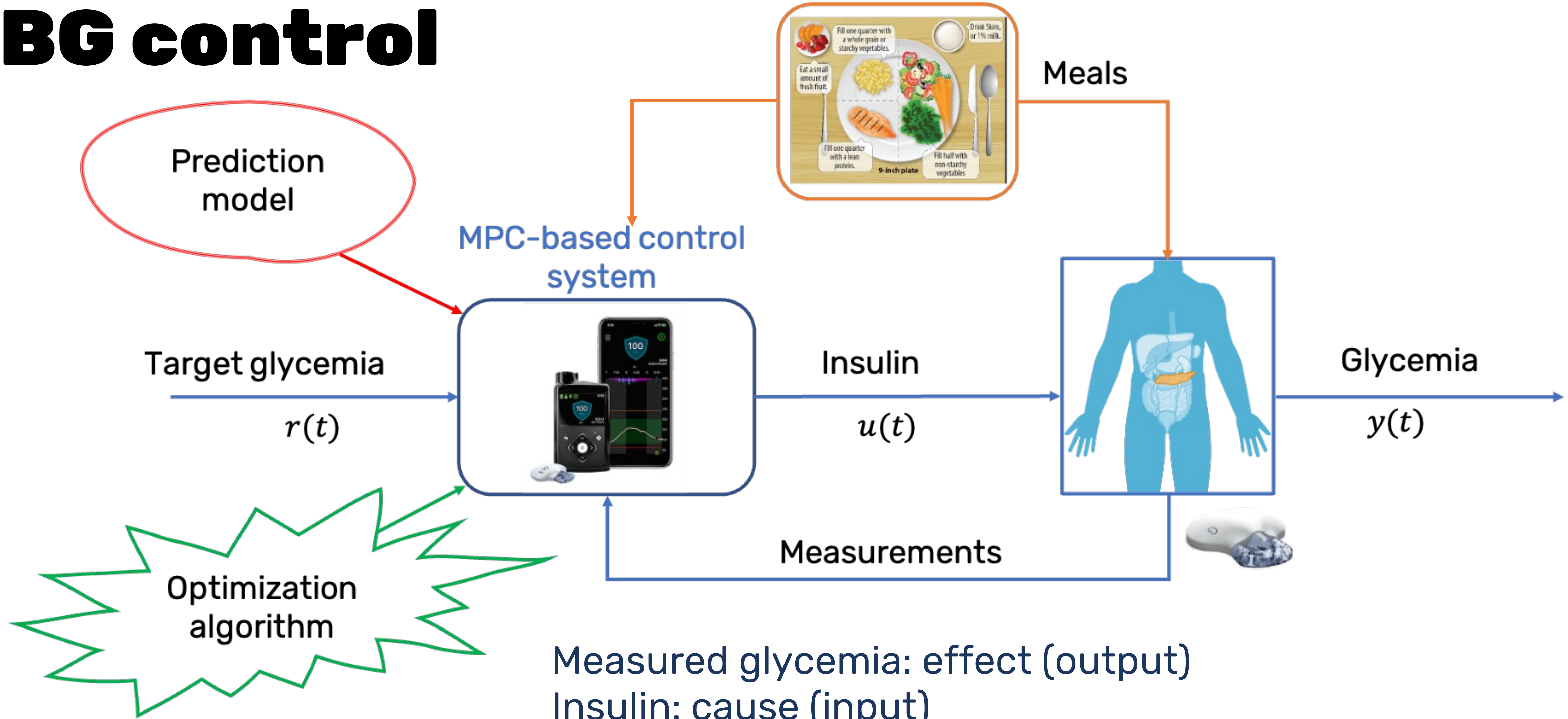
- **Outputs**: Total water flow and temperature.
- **Controlled inputs**: hot and cold water handles position
- **Disturbances**: inlet water flows, temperature

Drive a car…

Let’s suppose we want a model of a car for programming cruise control algorithm



BG control



Measured glycemia: effect (output)
Insulin: cause (input)
Meals: cause (disturbance)

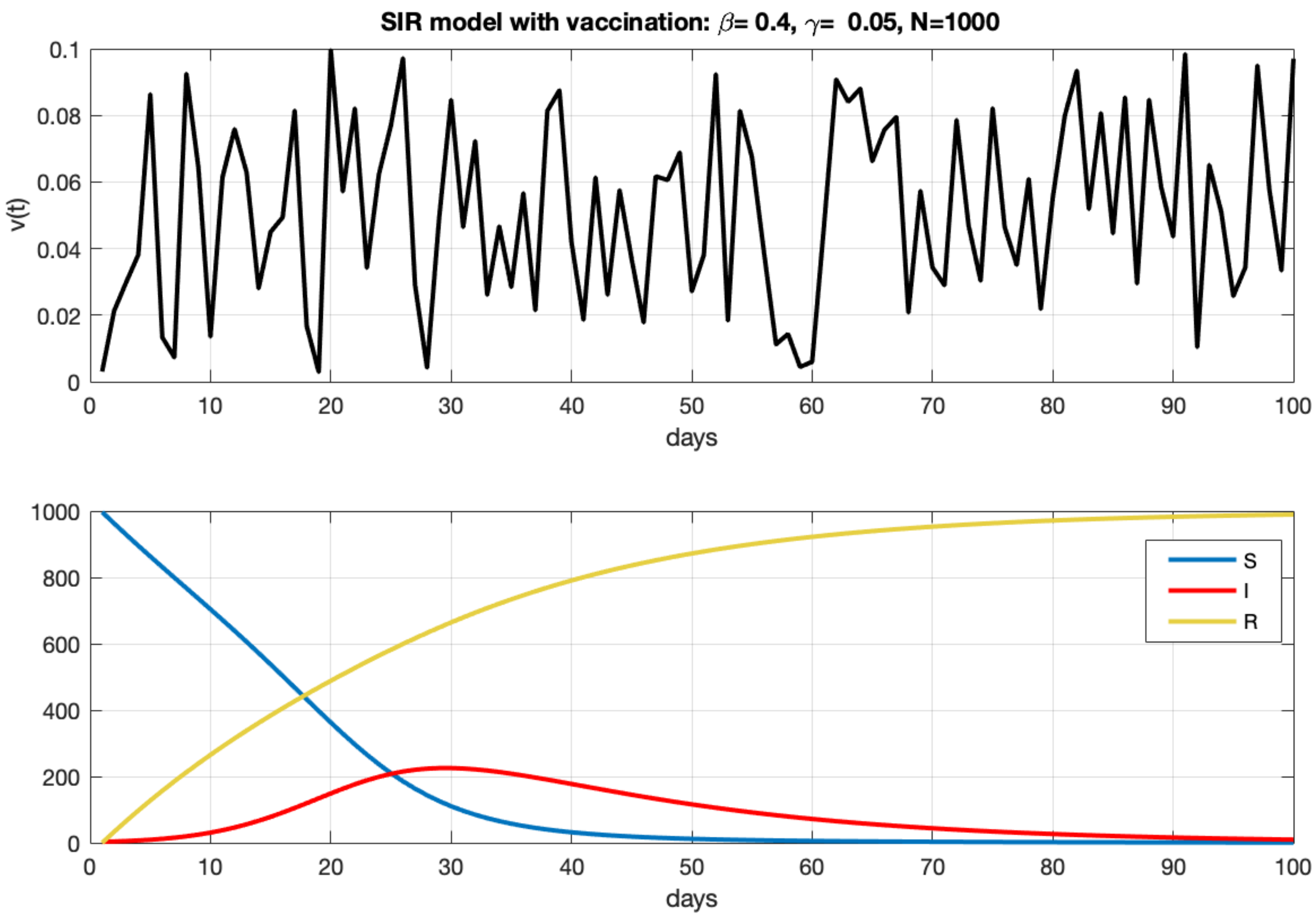
Example 3: SIR model with vaccination

- Let *v(t)* define the rate of susceptible subjects vaccinated at time *t*

$$\begin{aligned}
\dot{S}(t) &= -\frac{\beta S(t)I(t)}{N} - v(t)S(t) \\
\dot{I}(t) &= \frac{\beta S(t)I(t)}{N} - \gamma I(t) \\
\dot{R}(t) &= \gamma I(t) + v(t)S(t)
\end{aligned}$$

- v(t) manipulates the behavior of the SIR system

Example 3: SIR model with vaccination



- The number of infected is reduced by manipulating the number of suscetibles through vaccinations
- If v(t) is controlled by a user, it is a controlled input of the system

State variables

- SIR models are characterized by three main variables, *S(t), I(t), R(t),* which perfectly describe the phenomenon/system at a certain time *t.*

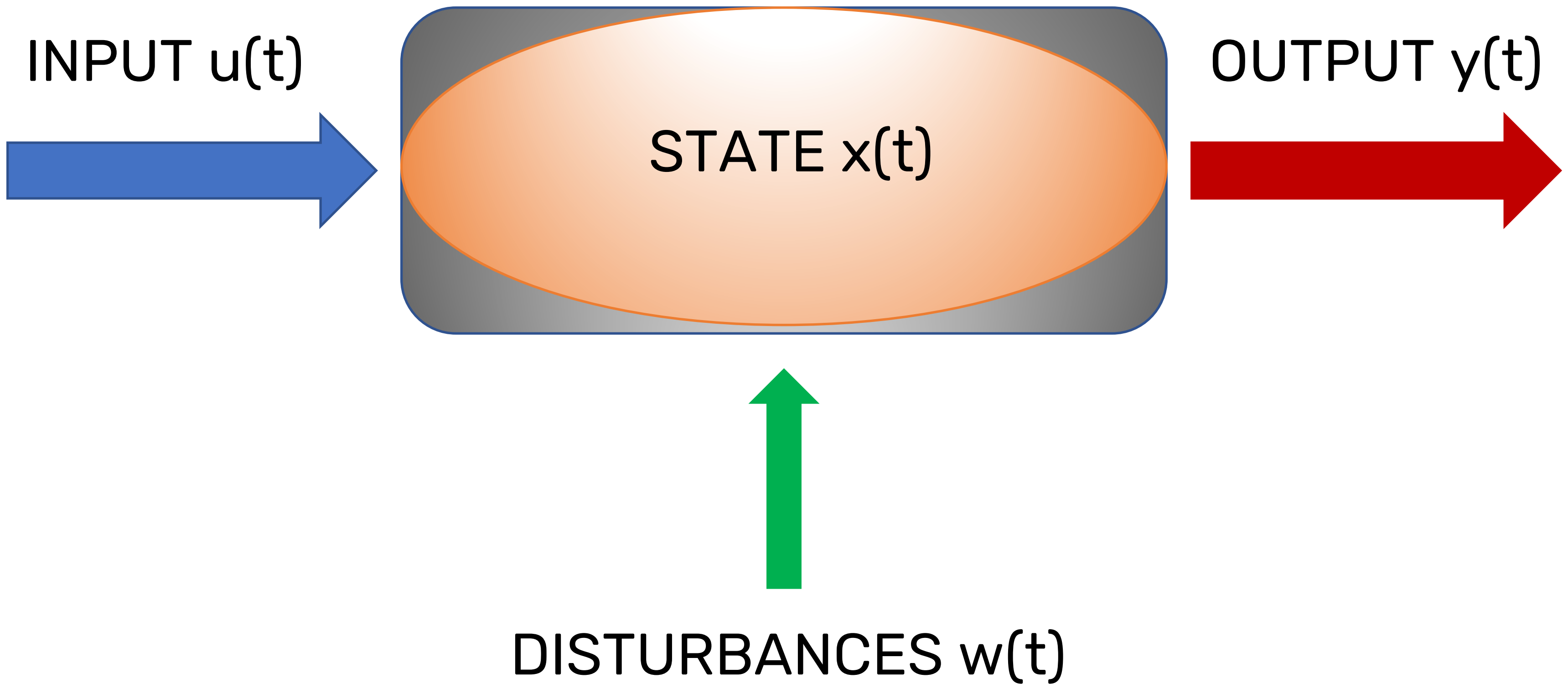
- These variables are called the **state of the system**.

- These variables are **internal variables** and may be infinite. They provide a **picture** of the system at time *t*.

- The state describes the **dynamic** of the system.

- State variables may or may not be measurable. In this last case, we can estimate the state (or a part of it) based on the ***known*** **input-output** model.

Graphical representation



Generic internal representation

A generic state-space continuous time model is described by the following set of equations:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

Where:

$x \in \mathbb{R}^n$ is the state of the system

$u \in \mathbb{R}^m$ is the input of the system

$y \in \mathbb{R}^p$ is the output of the system

$f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the state equation (a system of eq. in fact)

$g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is the output tranformation

Generic internal representation

- For our SIR model:

$$x(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}, \quad u(t) = v, \quad f(x(t), u(t)) = \begin{bmatrix} -\frac{\beta S(t) I(t)}{N} - v(t) S(t) \\ \frac{\beta S(t) I(t)}{N} - \gamma I(t) \\ \gamma I(t) + v(t) S(t) \end{bmatrix}$$

- We can also define an output *H(t)* as the number of known infected

$$y(t) = H(t), \quad g(x(t), u(t)) = aI(t) + b(S(t) + R(t))$$

Where *a* is the rate of the reported case and *b* is the rate of diagnosis errors.

Dynamical System Identification

Models that describe complex phenomena are not easy to be found.

To create a model a very **good knowledge of the problem** is required as well as a **good knowledge of the dynamical system theory**.

An alternative is to use specific sets of **input-output data** obtained from experiments, to infer the underline models by means of specific algorithms.

DYNAMICAL SYSTEM IDENTIFICATION

Outline

1. Introduction to dynamical systems

2. **Classification of dynamical systems**

What is a mathematical model?

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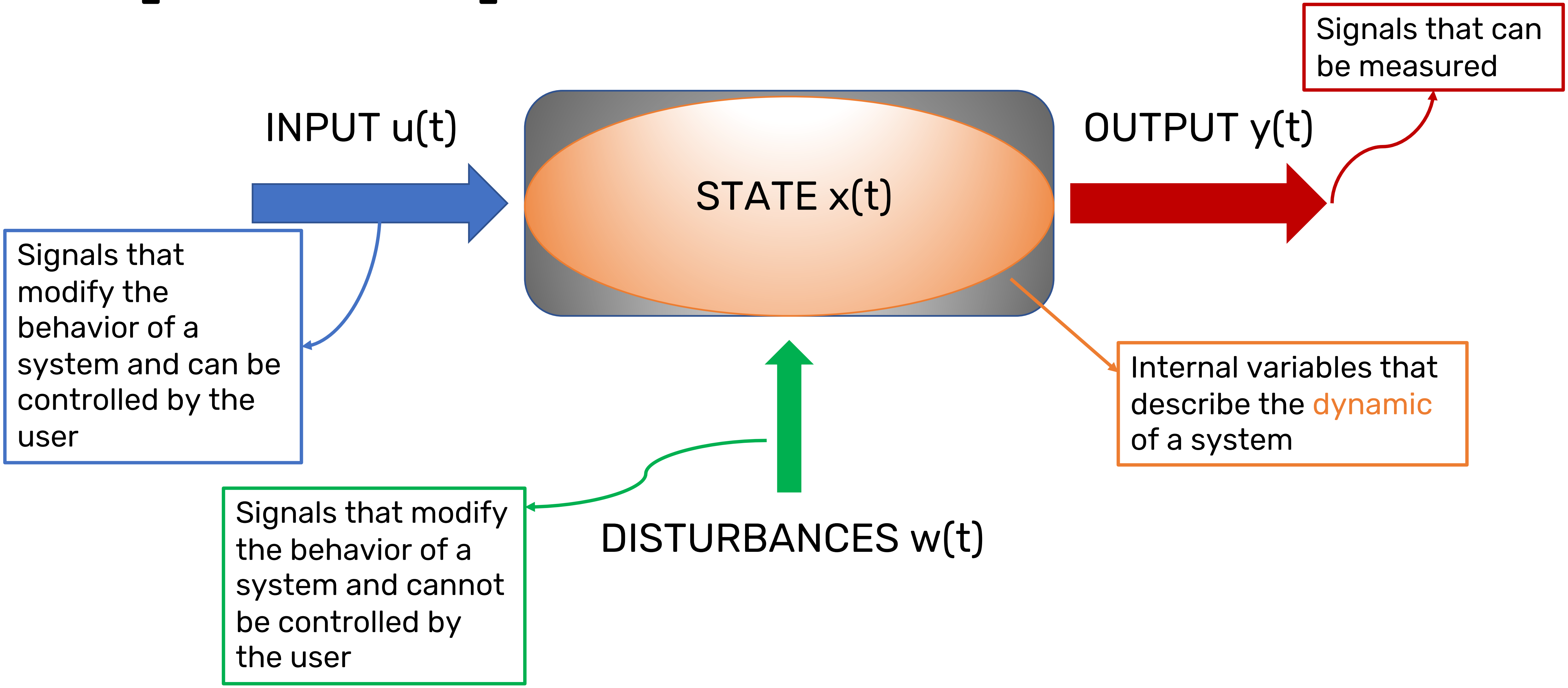
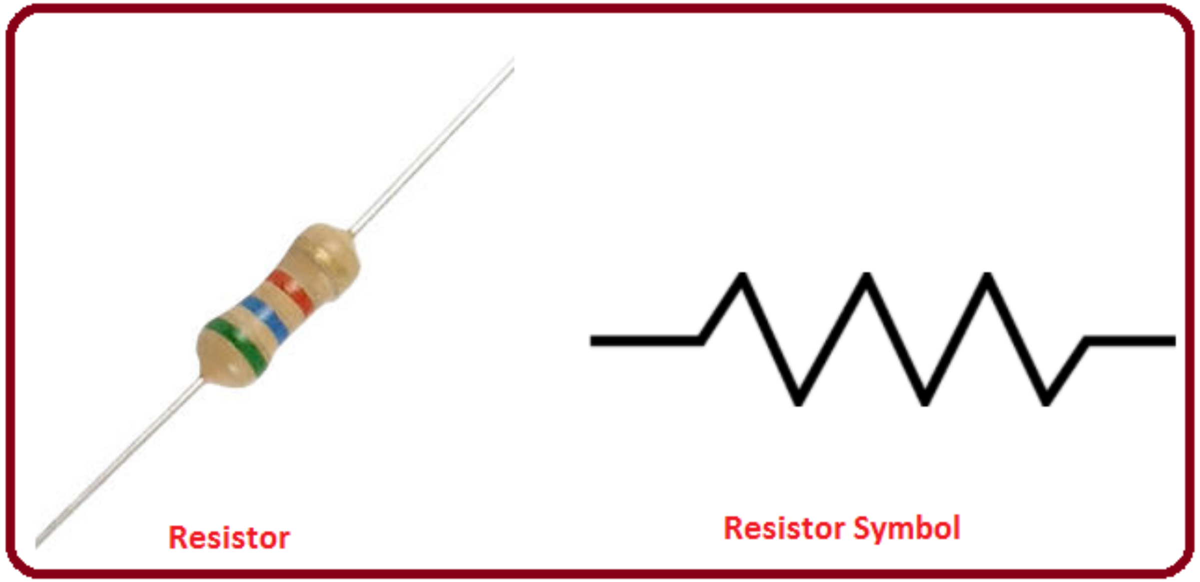
Graphical representation



Static vs dynamical systems

In a static system it is enough to know u(t) in order to compute y(t). The past has no effect on their evolution.



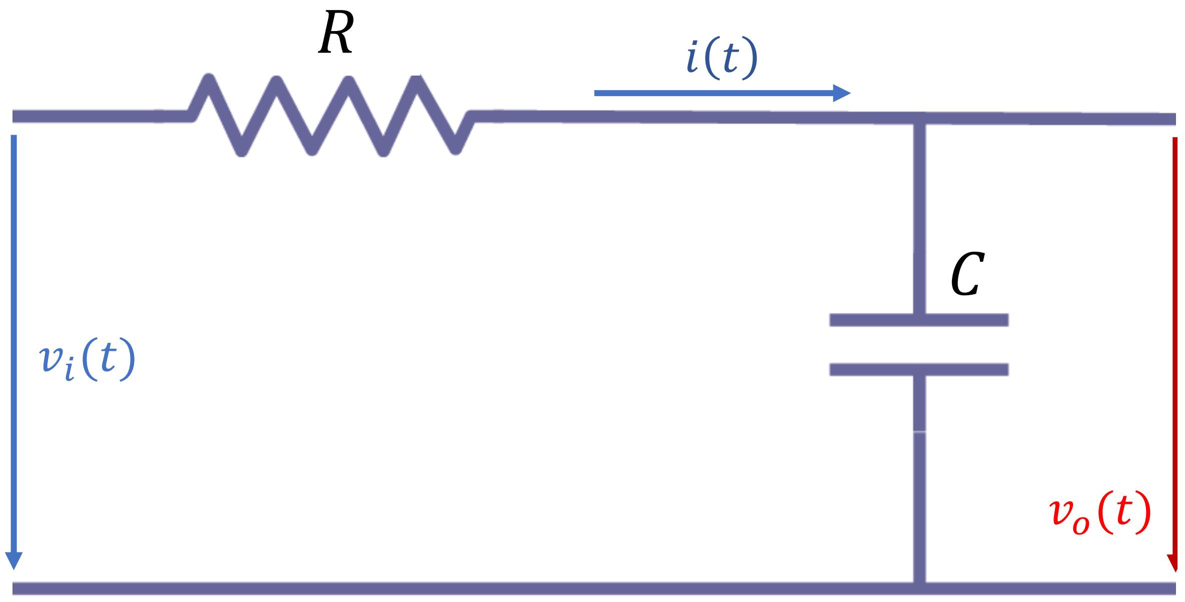
Ohm's law

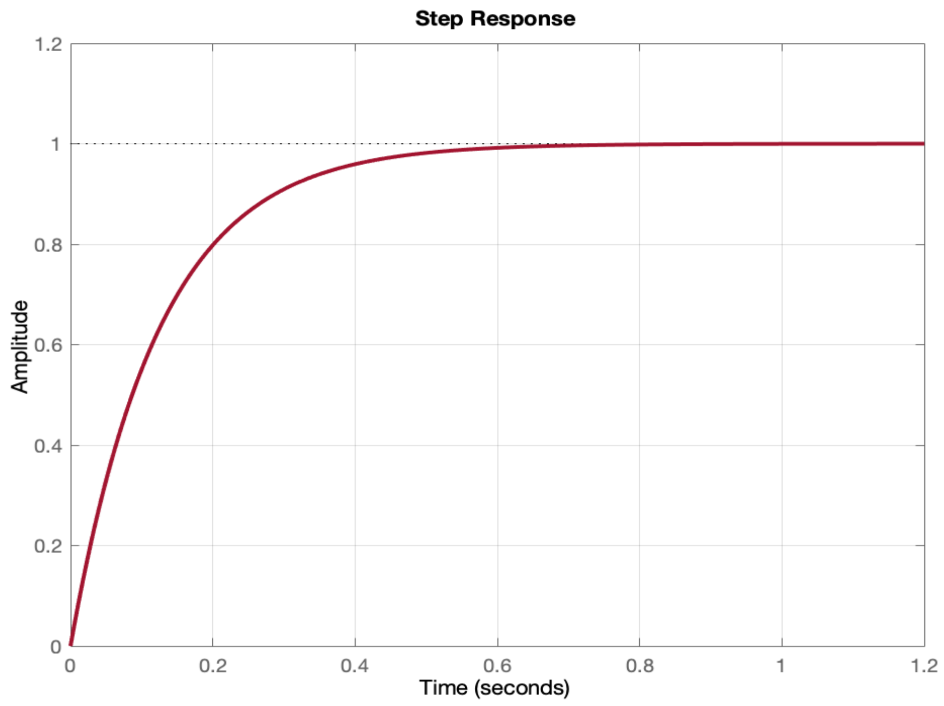
$$v(t) = Ri(t)$$

Given i(t) it is always possible to compute v(t)

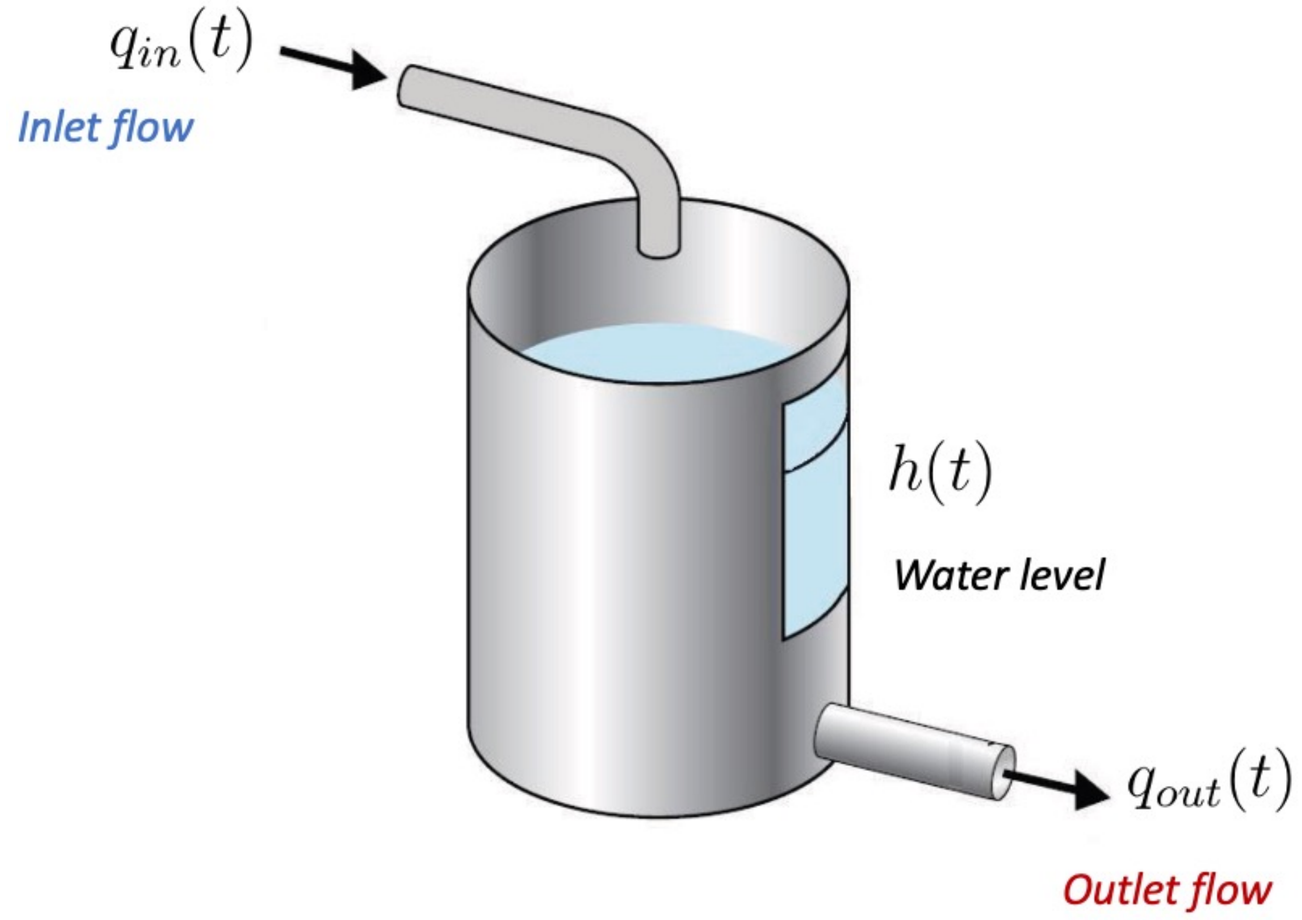
Static vs dynamical systems

In a dynamical system there is a sort of memory: one needs to know the past state in order to compute the new one.





Example: water tank



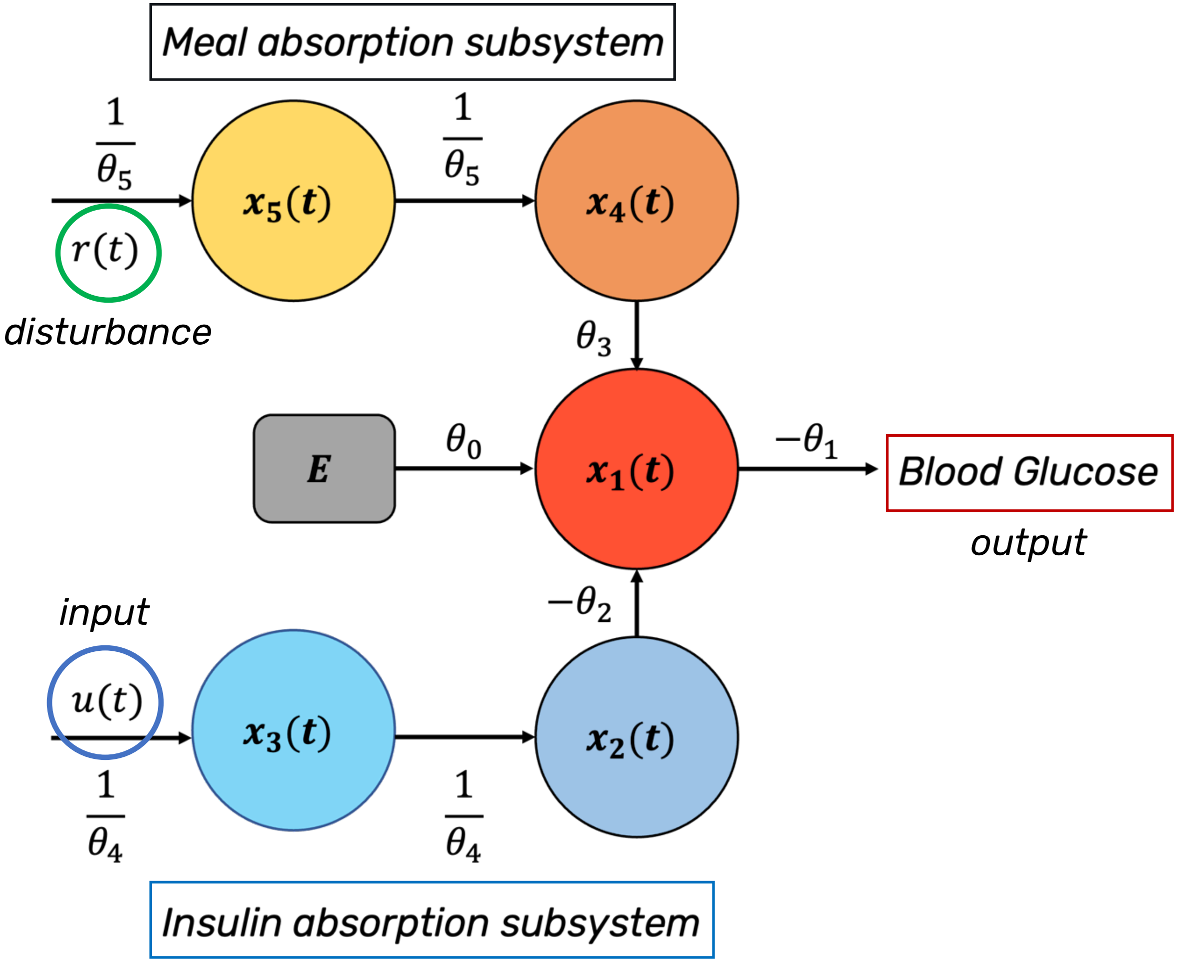
$$q_{in}(t) - q_{out}(t) = A\frac{dh(t)}{dt}$$

$$q_{out}(t) = \kappa\sqrt{h(t)}$$

$$\frac{dh(t)}{dt} = -\frac{\kappa\sqrt{h(t)}}{A} + \frac{q_{in}(t)}{A}$$

$h(t)$ State (internal variable) of the system

Example: blood glucose-insuline model



$$\frac{dQ_g(t)}{dt} = -\frac{1}{\theta_5} Q_g(t) + \frac{1}{\theta_5} Q_{sto}(t)$$

$$\frac{dQ_{sto}(t)}{dt} = -\frac{1}{\theta_5} Q_{sto}(t) + \frac{1}{\theta_5} r(t)$$

$$\frac{dG(t)}{dt} = \theta_0 - \theta_1 G(t) - \theta_2 Q_i(t) + \theta_3 Q_g(t)$$

$$\frac{dQ_i(t)}{dt} = -\frac{1}{\theta_4} Q_i(t) + \frac{1}{\theta_4} Q_{i_{sub}}(t)$$

$$\frac{dQ_{i_{sub}}(t)}{dt} = -\frac{1}{\theta_4} Q_{i_{sub}}(t) + \frac{1}{\theta_4} u(t)$$

State variables

In a dynamical system knowing $u(t)$ is not enough to determine $y(t)$.

A dynamical system has some kind of memory:

- the value of the output depends on the actual state
- The value of the actual state depends on the previous state.

A dynamical system is fully characterized by its input, its output and its state.

These are the **signals** of system.

State variables

The state variables $\boldsymbol{x}(t)$ are the internal variables whose initial condition $\boldsymbol{x}(t_0)$ represent the minimum information necessary to compute the output $\boldsymbol{y}(t)$ that corresponds to a certain input $\boldsymbol{u}(t)$.

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= g(x(t), u(t))
\end{aligned}
$$

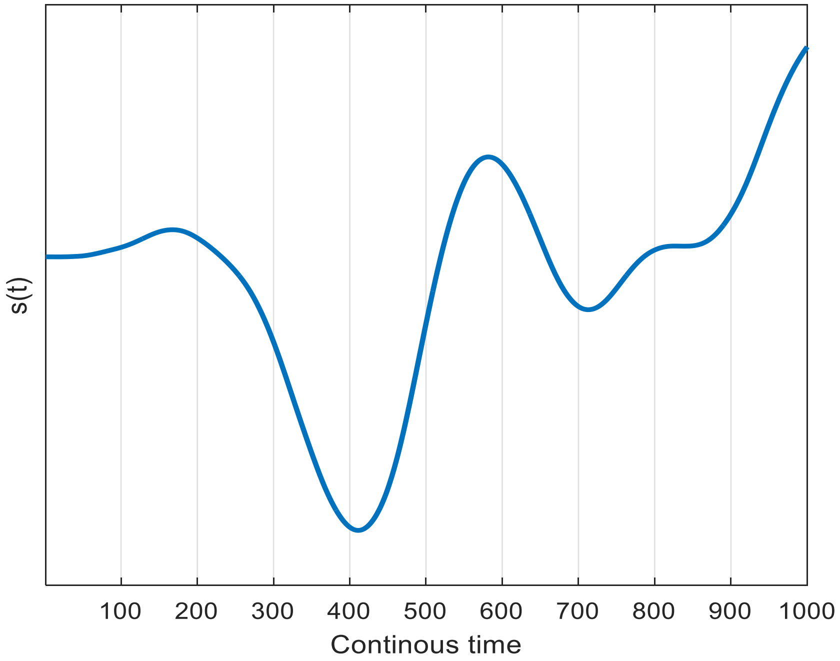
The dimension of the state n, i.e. $x \in \mathbb{R}^n$, defines the **order** of the system.

- For instance a SIR model is a system of order 3 since: $x(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}$

Continuous vs Discrete signals

A signal is a **real function** of time

Continuous signal

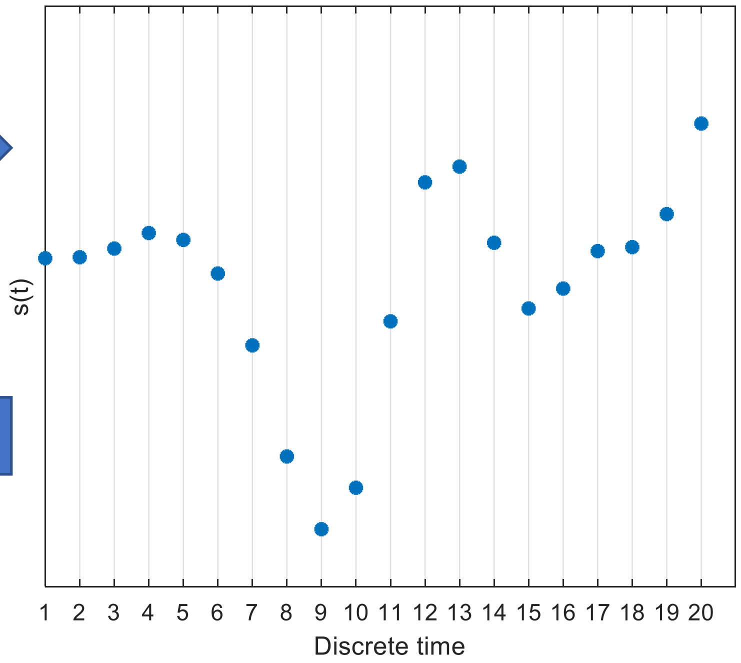


$$s : \mathbb{R} \rightarrow \mathbb{R}$$

Sampling

Reconstruction

Discrete signal

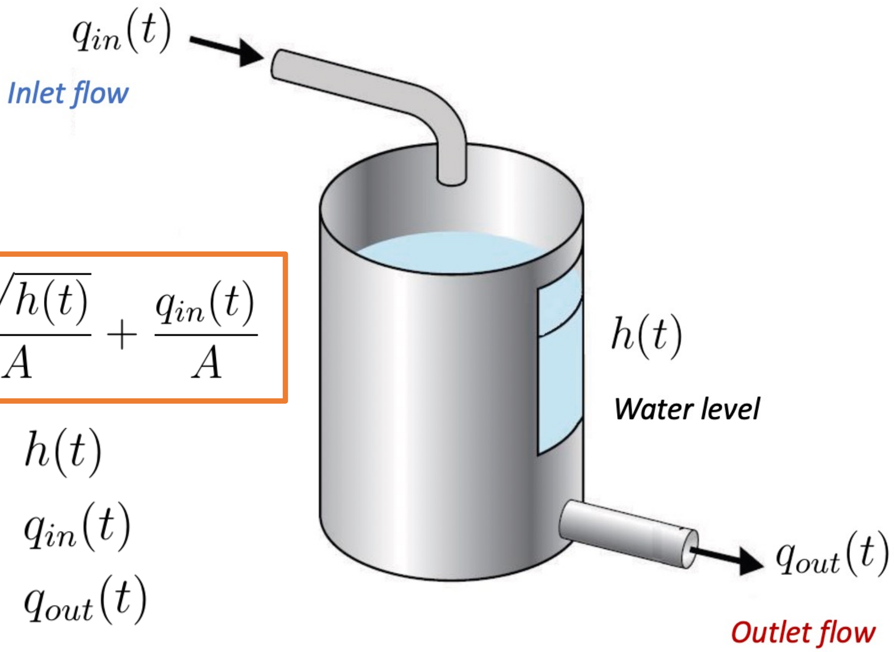


$$s : \mathbb{Z} \rightarrow \mathbb{R}$$

Only possible only in certain cases.

Continuous-time models

- States, inputs and outputs are continuous signals.
- Usually employed to represent physical system (reality is continuous).
- Representation: a set of differential equations



$$\frac{dh(t)}{dt} = -\frac{\kappa\sqrt{h(t)}}{A} + \frac{q_{in}(t)}{A}$$

$$\begin{aligned} x(t) &= h(t) \\ u(t) &= q_{in}(t) \\ y(t) &= q_{out}(t) \end{aligned}$$

$$\begin{aligned} \frac{dx(t)}{dt} &= -\frac{\kappa\sqrt{x(t)}}{A} + \frac{u(t)}{A} \\ y(t) &= \kappa\sqrt{x(t)} \end{aligned}$$

Output transformation

State equation

Discrete-time models

States, inputs and outputs are **discrete** signals.

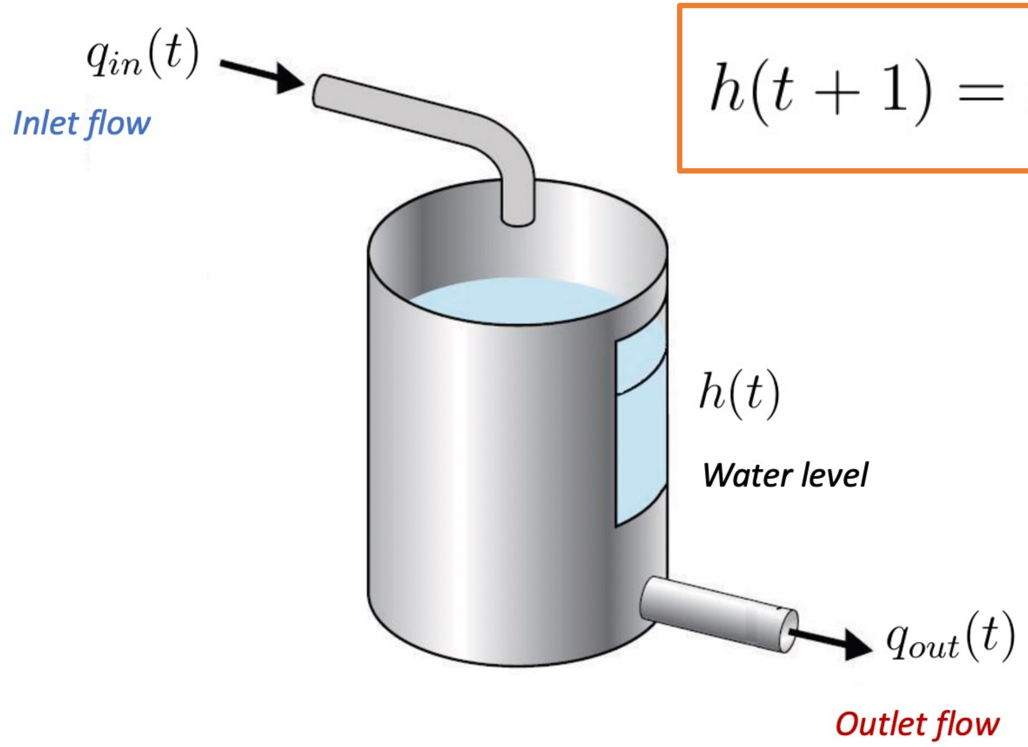
Most employed models in practice: **reality is continuous, but sensors/actuators work in discrete time.**

- ➢It is impossible to store an infinite number of data (continuous signal).

Representation: a set of finite-difference equations.

Also called digital models.

Discrete-time water tank



$$h(t+1) = h(t) + \frac{\Delta t}{A} \cdot (q_{in}(t) - q_{out}(t))$$

State: $x(t) = h(t)$
Input: $u(t) = q_{in}(t)$
Output: $y(t) = q_{out}(t)$

$$\begin{aligned} x(t+1) &= x(t) + \frac{\Delta t}{A}(u(t) - \kappa\sqrt{x(t)}) \\ y(t) &= \kappa\sqrt{x(t)} \end{aligned}$$

$$\begin{aligned} V(t+1) &= V(t) + q_{in}(t) \cdot \Delta t - q_{out}(t) \cdot \Delta t \\ A \cdot h(t+1) &= A \cdot h(t) + q_{in}(t) \cdot \Delta t - q_{out}(t) \cdot \Delta t \end{aligned}$$

Discrete-time models

- Discrete-time models are less general than the continuous one, because, typically, they strongly depend on the sampling interval Δt.

- There exist techniques that allow to go from continuous time to discrete time (with Δt known), but not vice versa.

- Both representations are important. The most useful one depends on the purpose of the modelization.

State-space representation

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

$$\begin{aligned} t &\in \mathbb{Z} \\ x(0) &= x_0 \end{aligned}$$

$x \in \mathbb{R}^n$ is the state of the system

$u \in \mathbb{R}^m$ is the input of the system

$y \in \mathbb{R}^p$ is the output of the system

$f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the state equation (a system of eq. in fact)

$g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is the output tranformation

n: order of the system

Classifications

A system is called **strictly proper** if the output transformation only depends on the value of the state. Otherwise, is said **proper**.

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

PROPER

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= g(x(t)) \end{aligned}$$

STRICTLY PROPER

Classifications

- A system is called **linear** if *f(.)* and *g(.)* are linear functions of the state and the input. Otherwise, is said **non-linear**.

- A system is called **time variant** if *f(.)* and *g(.)* depends explicitly on the value of *t*. Otherwise is said **time-invariant**.

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

$$\begin{aligned} x(t+1) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \end{aligned}$$

TIME-INVARIANT

TIME-VARIANT

Classifications

Last, but not least, we classify a system based on the dimension of its input and output

- A system is called **SISO (Single Input-Single Output)** if the input and the output are scalar, i.e. $u(t) \in \mathbb{R}, y(t) \in \mathbb{R}$
- A system is called **MISO (Multi Input-Single Output)** if the input dimension is >1 and the output is scalar, i.e. $u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}, m > 1$
- A system is called **SIMO (Single Input-Multi Output)** if the input is scalar and the output dimension is >1, i.e. $u(t) \in \mathbb{R}, y(t) \in \mathbb{R}^p, p > 1$
- A system is called **MIMO (Multi Input - Multi Output)** if the input and output dimensions are >1, i.e. $u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, m > 1, p > 1$

Example 1: water tank

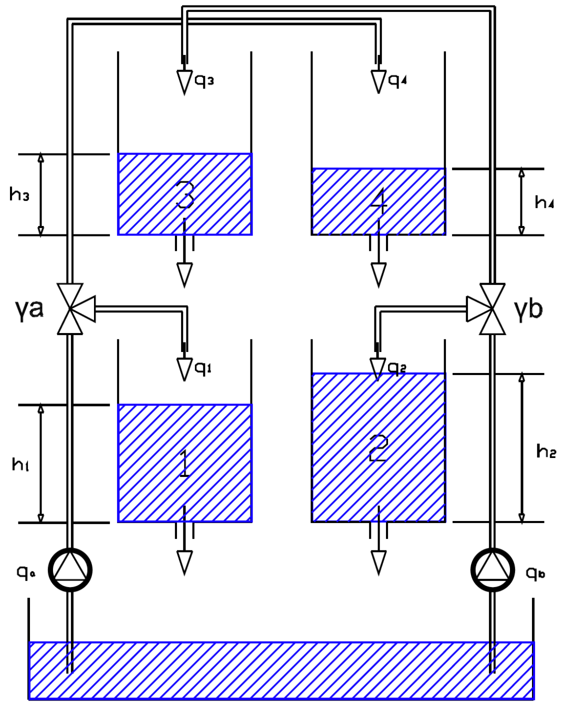
$$
\begin{aligned}
x(t+1) &= x(t) + \frac{\Delta t}{A}\left(u(t) - \kappa\sqrt{x(t)}\right) \\
y(t) &= \kappa\sqrt{x(t)}
\end{aligned}
$$

- The system is nonlinear.
- The system is time-invariant.
- Order: 1 (only one state)
- SISO system: only one input and one output
- Strictly proper: the output transformation does not depend on u(t)

Example 2: 4 water tanks

$$\mathbf{x}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ h_4(t) \end{bmatrix} \quad \mathbf{u}(t) = \begin{bmatrix} q_a(t) \\ q_b(t) \end{bmatrix} \quad \mathbf{y}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$$

$$\begin{aligned}
h_1(t+1) &= h_1(t) - \frac{\Delta t}{S}\left(a_1\sqrt{2gh_1(t)} + a_3\sqrt{2gh_3(t)} + \gamma_a q_a(t)\right) \\
h_2(t+1) &= h_2(t) - \frac{\Delta t}{S}\left(a_2\sqrt{2gh_2(t)} + a_4\sqrt{2gh_4(t)} + \gamma_b q_b(t)\right) \\
h_3(t+1) &= h_3(t) - \frac{\Delta t}{S}\left(a_3\sqrt{2gh_3(t)} + (1-\gamma_b) q_b(t)\right) \\
h_4(t+1) &= h_4(t) - \frac{\Delta t}{S}\left(a_4\sqrt{2gh_4(t)} + (1-\gamma_a) q_a(t)\right)
\end{aligned}$$



- The system is nonlinear.
- The system is time-invariant.
- Order: 4
- MIMO system: 2 input and 2 output
- Strictly proper: the output transformation does not depend on u(t)

Example 3: SIR with vaccination

$$
\begin{aligned}
S(t+1) &= S(t) - \frac{\beta S(t) I(t)}{N} - v(t) S(t) \\
I(t+1) &= I(t) + \frac{\beta S(t) I(t)}{N} - \gamma I(t) \\
R(t+1) &= R(t) + \gamma I(t) + v(t) S(t) \\
y(t) &= = aI(t) + b(S(t) + R(t))
\end{aligned}
$$

$$
\mathbf{x}(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}, \quad u(t) = v(t),
$$

- The system is nonlinear.
- The system is time-invariant.
- Order: 3
- SISO system: only one input and one output
- Strictly proper: the output transformation does not depend on u(t)

Example 4: bg-insuline model

$x_1(t)$ $$\frac{dG(t)}{dt} = \theta_0 - \theta_1 G(t) - \theta_2 Q_i(t) + \theta_3 Q_g(t)$$

$x_2(t)$ $$\frac{dQ_i(t)}{dt} = -\frac{1}{\theta_4} Q_i(t) + \frac{1}{\theta_4} Q_{i_{sub}}(t)$$

$x_3(t)$ $$\frac{dQ_{i_{sub}}(t)}{dt} = -\frac{1}{\theta_4} Q_{i_{sub}}(t) + \frac{1}{\theta_4} u(t)$$

$x_4(t)$ $$\frac{dQ_g(t)}{dt} = -\frac{1}{\theta_5} Q_g(t) + \frac{1}{\theta_5} Q_{sto}(t)$$

$x_5(t)$ $$\frac{dQ_{sto}(t)}{dt} = -\frac{1}{\theta_5} Q_{sto}(t) + \frac{1}{\theta_5} r(t)$$

- The system is linear.
- The system is time invariant.
- Order: 5
- MISO system: 2 inputs and 1 output
- Strictly Proper: the output transformation does not depend on u(t)

$y(t) = G(t)$

$u(t) = insuline\ infusion\ (controllable\ input)$

$r(t) = meals\ (non - controllable\ input, disturbance)$

Example 5

$$
\begin{aligned}
x_1(t+1) &= 2x_2(t) - u(t) \\
x_2(t+1) &= x_1(t) \\
y(t) &= x_1(t) + t \cdot u(t)
\end{aligned}
$$

- The system is linear.
- The system is time variant.
- Order: 2
- SISO system: only one input and one output
- Proper: the output transformation depends on u(t)

State variables

- The number of states variables n determines the **complexity of the system.**
  - A more complex system can model the same phenomena more accurately.

- With a fixed complexity the order of the system is uniquely determined.
- There are different choices of the state variables that define the same input-output relation, even if the order is always the same.
- The minimum number of states is called **minimal representation**, and equals the number of differential (difference) equations describing the phenomenon.
- Sometimes the context provides criteria (physic, for example) for the selection of the state variables.

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