Discrete-time dynamic systems

Solutions

Exercise 1

Consider the system given by:

$$\begin{aligned} x_1(k+1) &= 0.6x_1(k) + u(k) \\ x_2(k+1) &= 0.4x_2(k) + 5u(k) \\ y(k) &= x_1(k) + 3x_2(k) \end{aligned}$$

1. Classify the system.
2. Find the state-space matrices of the system.
3. Determine the stability of the system.
4. Find the transfer function of the system, assuming $y(0) = 0$.
5. Determine the static gain of the system.

Solution

1. The system is an LTI discrete time system, second order. SISO, strictly proper.
2. The state-space matrices of the system are

$$A = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 \end{bmatrix}, D = 0$$

3. The system is asymptotically stable if all eigenvalues of A are inside the unit circle. In this case, A is a diagonal matrix, so its eigenvalues are the element of its diagonal

$$\lambda_{1,2} = \{0.6, 0.4\}, |\lambda_i| < 1$$

So the system is asymptotically stable.

4. Since $y(0) = 0$, the transfer function can be found by applying the formula

$$G(z) = C(zI - A)^{-1}B + D$$

Taking into account the values of (A, B, C, D) found in the second point, we get:

$$G(z) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda - 0.6 & 0 \\ 0 & \lambda - 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 0$$

Then:

$$G(z) = \frac{16z - 9.4}{z^2 - z + 0.24}$$

5. The static gain of the system is given by $\mu = G(1)$, then

$$G(1) = \frac{16 - 9.4}{1 - 1 + 0.24} = 27.5$$

Exercise 2

Consider the system given by:

$$\begin{aligned} x(k+1) &= (\alpha^2 - 3\alpha)x(k) + \beta u(k) \\ y(k) &= x(k) \end{aligned}$$

1. Find the state-space representation for the dynamics of the system and classify the system.
2. Determine for which values of α and β the systems is asymptotically stable.
3. Find the transfer function of the system, assuming $y(0) = 0$.
4. Determine the static gain of the system.

Solution

1. The system is an LTI discrete time system, first order. SISO, strictly proper. The system is already in state-space, and the matrices of such representation are:

$$A = (\alpha^2 - 3\alpha),\ B = \beta,\ C = 1,\ D = 0$$

2. Stability only depends on matrix A, so in this case only on α. In particular it has to be fulfilled that

$$|\alpha^2 - 3\alpha| < 1$$

This implies the following two conditions:

$$\alpha^2 - 3\alpha < 1 \Rightarrow \alpha^2 - 3\alpha - 1 < 0$$

which is true for

$$\frac{3-\sqrt{13}}{2} < \alpha < \frac{3+\sqrt{13}}{2}$$

And

$$\alpha^2 - 3\alpha > -1 \Rightarrow \alpha^2 - 3\alpha + 1 > 0$$

which is true for

$$\alpha < \frac{3-\sqrt{5}}{2} \text{ and } \alpha > \frac{3+\sqrt{5}}{2}$$

Since the previous conditions have to be all true at the same time, the we can conclude that the system is Asymptotically stable for any value of α such that

$$\frac{3-\sqrt{13}}{2} < \alpha < \frac{3-\sqrt{5}}{2},\ \text{AND}\ \frac{3+\sqrt{5}}{2} < \alpha < \frac{3+\sqrt{13}}{2}$$

The system is simply stable if

$$|\alpha^2 - 3\alpha| = 1$$

which occurs if α takes one of the following values:

$$\alpha = \{\frac{3-\sqrt{13}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{13}}{2}\}$$

The system is unstable if

$$|\alpha^2 - 3\alpha| > 1$$

which occurs if α takes values inside the following intervals:

$$\alpha < \frac{3-\sqrt{13}}{2} \text{ OR } \frac{3-\sqrt{5}}{2} < \alpha < \frac{3+\sqrt{5}}{2} \text{ OR } \alpha > \frac{3+\sqrt{13}}{2}$$

3. Since $y(0) = 0$, the transfer function can be found by applying the formula

$$G(z) = C(zI - A)^{-1}B + D$$

Taking into account the values of (A, B, C, D) found in the first point, we get:

$$G(z) = \frac{\beta}{z - \alpha^2 + 3\alpha}$$

4. The static gain of the system is given by $\mu = G(1)$, then

$$G(1) = \frac{\beta}{1 - \alpha^2 + 3\alpha}$$

Exercise 3

Consider the following matrices of a state-space representation of a certain system:

$$A = \begin{bmatrix} -0.74 & 0.56 \\ -0.26 & 0.44 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad D = 0$$

1. Find the state-space difference equation of the system.
2. Determine the stability of the system
3. Find the transfer function of the system, assuming $y(0) = 0$ and check its poles.
4. Determine the static gain of the system.

Solution

1. The difference equation are given by

$$\begin{aligned} x_1(t+1) &= -0.74x_1(t) + 0.56x_2(t) + 0.5u(t) \\ x_2(t+1) &= -0.26x_1(t) + 0.44x_2(t) + u(1) \\ y(t) &= 0.5x_1(t) + 0.5x_2(t) \end{aligned}$$

2. The stabiliy of the systems depends on the eigenvalues of A. Such eigenvalues are given by

$$\lambda_{1,2} = \{-0.6, 0.3\}$$

which are both inside the unit circle. So the system is asymptotically stable.

3. Since $y(0) = 0$, the transfer function can be found by applying the formula

$$G(z) = C(zI - A)^{-1}B + D$$

Taking into account the values of (A, B, C, D), we get:

$$G(z) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \lambda + 0.74 & -0.56 \\ 0.26 & \lambda - 0.44 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} + 0$$

Then:

$$G(z) = \frac{0.75z - 0.25}{z^2 + 0.3z - 0.18}$$

The poles of $G(z)$ are the roots of its denominator. Then

$$z^2 + 0.3z - 0.18 = 0 \Rightarrow p_{1,2} = \{-0.6, 0.3\}$$

which are exactly the same as the eigenvalues of A.

4. The static gain of the system is given by $\mu = G(1)$, then

$$\mu = G(1) = \frac{0.75 - 0.25}{1 + 0.3 - 0.18} = 0.4464$$

Exercise 4

Consider the system given by:

$$\begin{aligned} x_1(k+1) &= x_1(k) + (1 - x_1(k))(1 + x_2(k)) + u(k) \\ x_2(k+1) &= x_1(k) + (1 + x_1(k))(1 - x_2(k)) - u(k) \\ y(k) &= (x_1(k) + x_2(k))^3 \end{aligned}$$

1. Find the equilibrium states and output for $u(k) = \bar{u} = 0$.
2. Find the linearized systems about such equilibria.
3. Determine the stability of the linearized systems.

Solution

1. The equilibrium states are such that $x_i(k+1) = x_i(k) = \bar{x}_i$, $i = 1, 2$. Then:

$$\begin{aligned} \bar{x}_1 &= \bar{x}_1 + (1 - \bar{x}_1)(1 + \bar{x}_2) + \bar{u} \\ \bar{x}_2 &= \bar{x}_1 + (1 + \bar{x}_1)(1 - \bar{x}_2) - \bar{u} \\ \bar{y} &= (\bar{x}_1 + \bar{x}_2)^3 \end{aligned}$$

   Taken into account that $\bar{u} = 0$, then from the first equation we get

$$(1 - \bar{x}_1)(1 + \bar{x}_2) = 0 \Rightarrow 1 + \bar{x}_2 - \bar{x}_1 - \bar{x}_1\bar{x}_2 = 0 \Rightarrow \bar{x}_1 = 1$$

   From the second equation:

$$\bar{x}_2 = 1 + (1 + 1)(1 - \bar{x}_2) = 0 \Rightarrow 3\bar{x}_2 = 3 \Rightarrow \bar{x}_2 = 1$$

   And finally

$$\bar{y} = (1 + 1)^3 = 8$$

2. The linearized system is obtained by taking the partial derivative of the system equations w.r.t. (x_1, x_2, u, y) and evaluating them in the equilibrium point. Then:

$$\begin{aligned} \delta x_1(k+1) &= -\bar{x}_2 \delta x_1(k) + (1 - \bar{x}_1)\delta x_2(k) + \delta u(k) \\ \delta x_2(k+1) &= (2 - \bar{x}_2)\delta x_1(k) + (-\bar{x}_1 - 1)\delta x_2(k) - \delta u(k) \\ \delta y(k) &= 3(\bar{x}_1 + \bar{x}_2)^2 \delta x_1(k) + 3(\bar{x}_1 + \bar{x}_2)^2 \delta x_2(k) \end{aligned}$$

   And by substituting the values obtained in the previous point, we get

$$\begin{aligned} \delta x_1(k+1) &= -\delta x_1(k) + \delta u(k) \\ \delta x_2(k+1) &= \delta x_1(k) - 2\delta x_2(k) - \delta u(k) \\ \delta y(k) &= 12\delta x_1(k) + 12\delta x_2(k) \end{aligned}$$

   The state-space matrices of the system are

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 12 & 12 \end{bmatrix}, D = 0$$

3. To asses the stability of the equilibrium point, it is enough to look at the eigenvalues of the matrix A of the linearized system. We get

$$\lambda_{1,2} = \{-1, -2\}$$

   which are one outside and one on the boundary of the unit circle. So the equilibrium point is unstable.

Exercise 5

Consider the system given by:

$$\begin{aligned} x(k+1) &= x^2(k) + u(k) \\ y(k) &= x(k)u(k) \end{aligned}$$

1. Determine the equilibrium states of the system, as a function of the equilibrium input $\bar{u}$.
2. Determine the stability of the system in the equilibrium point given by $\bar{u} = 1$.

Solution

1. The equilibrium states are such that $x(k+1) = x(k) = \bar{x}$. Then:

$$\bar{x} = \bar{x}^2 + \bar{u}$$

Then,

$$\bar{x}^2 - \bar{x} + \bar{u} = 0, \Rightarrow \bar{x}_{1,2} = \frac{1 \pm \sqrt{1-4u}}{2}$$

which are 2 possible equilibrium points.

2. For $\bar{u} = 1$, we get

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

In order to assess the stability of these equilibria, we can use the *indirect Lyapunov method*: we will find the linearized system about these equilibria and study the stability of the linearized systems.

$$\begin{aligned} \delta x(k+1) &= 2\bar{x}\delta x(k) + \delta u(k) \\ \delta y(k) &= \bar{u}\delta x(k) + \bar{x}\delta u(k) \end{aligned}$$

For $\bar{x} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ we have

$$\begin{aligned} \delta x(k+1) &= (1 + i\sqrt(3))\delta x(k) + \delta u(k) \\ \delta y(k) &= \delta x(k) + (1 + i\sqrt(3))\delta u(k) \end{aligned}$$

Matrix A for this system is given by $A = (1 + i\sqrt(3))$ which is a complex number with module $|A| = 2$, which is grater than 1. So the equilibrium $\bar{x} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is unstable.

As for $\bar{x} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ we have

$$\begin{aligned} \delta x(k+1) &= (1 - i\sqrt(3))\delta x(k) + \delta u(k) \\ \delta y(k) &= \delta x(k) + (1 - i\sqrt(3))\delta u(k) \end{aligned}$$

Matrix A for this system is given by $A = (1 - i\sqrt(3))$ which is a complex number with module $|A| = 2$, which is grater than 1. So the equilibrium $\bar{x} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ is unstable.